Statistics, Distributions and Probability

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Saha Institute of Nuclear Physics
Kolkata

Why Statistics?

What is Statistics?

- As a discipline
- As a quantifier
- Data Modelling

Probability

- Histograms as Probability Distribution
- Probability Theory

Need for Statistics

It is often said that the language of science is mathematics. It could well be said that the language of experimental science is statistics. It is through statistical concepts that we quantify the correspondence between theoretical predictions and experimental observations.

Kyle Cranmer, `1503.07622`

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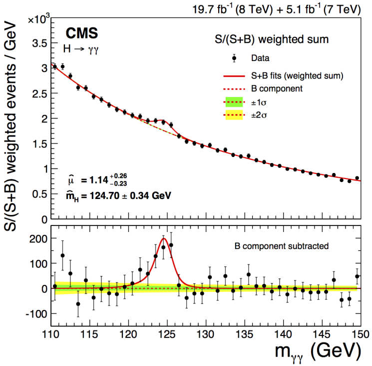
Positive outcome

Peak in invariant mass distribution of two photons $\Rightarrow$

Higgs mass (in GeV) = $125.09 \pm 0.21(stat.) \pm 0.11(syst.)$ (ATLAS + CMS)



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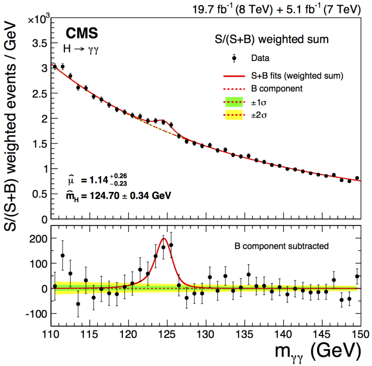

- Parameter estimation
- Hypothesis testing

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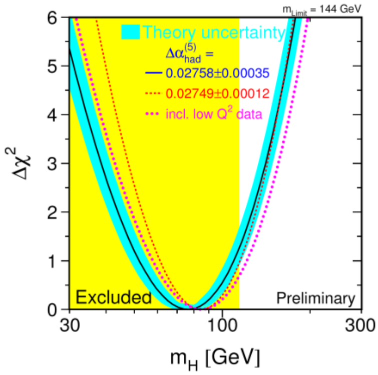
No conclusive evidence of signals of Higgs boson at LEP ⇒

$m_H > 114.3$ GeV at 95% C.L.
Best fit for m_H
Upper limit on m_H (SM like)



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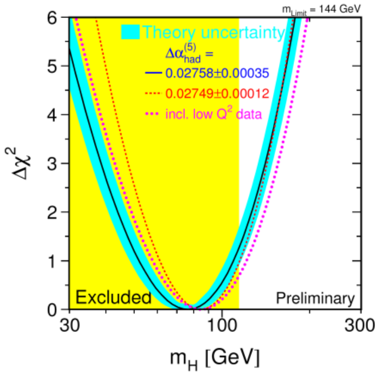

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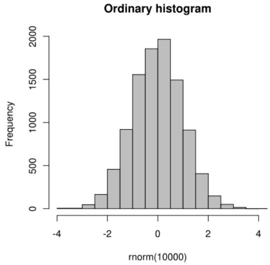
- drawing conclusions from data (translating quantifiers into model parameters),
- estimating the present or predicting the future (post-diction) or (pre-diction)

Statistics ≡ quantifiers

Let $\mathcal{D} = \{x_1, x_2, \ldots, x_N\}$ be a large set of data.

- Histogram, frequency distribution



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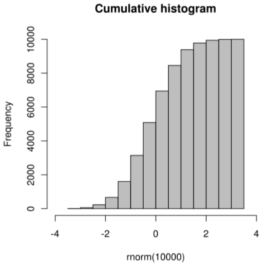

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Let $\mathcal{D} = \{x_1, x_2, \ \dots \ , x_N\}$ be a large set of data.

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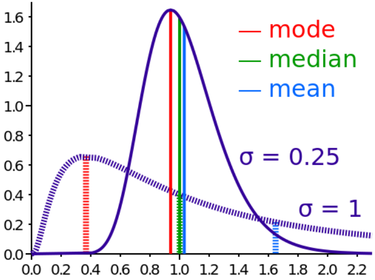
- Cumulative distribution,
- Median center of distribution
- Mode, most frequent
- Mean, center of mass



Moments of distribution

Mean

Center of mass for a given data set

$$\langle x \rangle = \frac{1}{N} \sum_{i=1}^{N} x_i$$

Mean for a given histogram

$$\langle x \rangle = \left(\sum_{i=1}^{H} w_i \right)^{-1} \sum_{i=1}^{H} w_i \ x_i$$

Here H is number of histograms and w_i is the frequency in the i^{th} histogram.

Higher Moments

For a given data set n^{th} moment is:

$$\langle x^n \rangle = \frac{1}{N} \sum_{i=1}^{N} x_i^n$$

and for a given histogram is:

$$\langle x^n \rangle = \left(\sum_{i=1}^{H} w_i \right)^{-1} \sum_{i=1}^{H} w_i \ x_i^n$$

The mean is the first ($n = 1$) moment. Other important quantifier is related to $\langle x^2 \rangle$.

Moments of distribution

Variance

Mean or average, $\langle x \rangle$ is a measure of center of the data. Variance measures the spread of data around the mean. Since we have:

$$\langle (x - \langle x \rangle) \rangle = 0$$

The variance is defines as

$$V = \langle (x - \langle x \rangle)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$$

Higher Moments

Show:

- $\langle (x - \langle x \rangle)^3 \rangle = \langle x^3 \rangle + 2\langle x \rangle^3 - 3\langle x \rangle \langle x^2 \rangle$
  Skewness
- $\langle (x - \langle x \rangle)^4 \rangle = \langle x^4 \rangle - 3\langle x \rangle^4 + 6\langle x^2 \rangle \langle x \rangle^2 - 4\langle x \rangle \langle x^3 \rangle$
  Kurtosis

Show these relations numerically during Tutorials.

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- Chosing H number of bins, the data set can be converted into a histogram (frequency distribution). Now a set of N data points are represented or modelled by set of $2H$ number, H for the bin centers and H for frequencies. This is the first step of modelling, where we replace a large set of data by its frequency distribution.

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- Example: If the shape of the histogram is Gaussian, it is parametrized with the mean and the variance.

$$\mathcal{D} \Rightarrow \langle x \rangle \pm \sqrt{V} = \langle x \rangle \pm \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

Histogram ≡ Probabilities

A group of 50 people has following age distribution

Age	#
0 – 5	2
5 – 10	8
10 – 15	15
15 – 20	14
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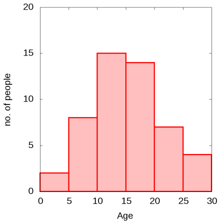

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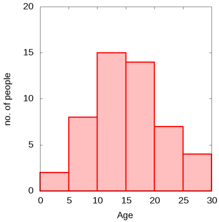

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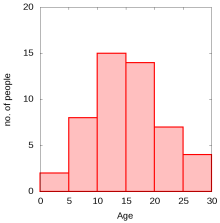

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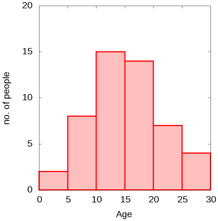

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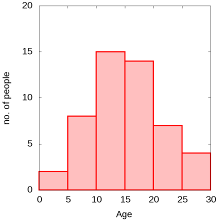

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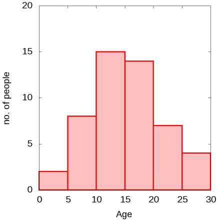

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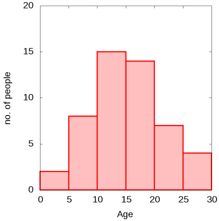
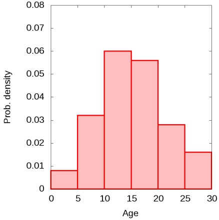
Probability density = probability/(bin width)





Probability Density Functions

- If $p(x)$ is the PDF of x then $p(x)dx$ is the probability of finding the value between x and $x + dx$.
- $\int_R dx\ p(x) = 1$, R is the range of variable x.
- $p(x) \geq 0$ for $x \in R$

Uniform (x, a, b), $x \in [a, b]$

$$\frac{1}{b-a}$$

Exponential (x, a), $x \in [0, \infty]$

$$\frac{1}{a} \exp(-x/a)$$

Gaussian (x, μ, σ), $x \in [-\infty, \infty]$

$$\frac{1}{\sqrt{2\pi\sigma}} \exp\left[\frac{(x-\mu)^2}{2\sigma^2}\right]$$

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Conditional Probability

If A and B are independent then probability of A and B

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If A and B are independent

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = P(A)$$

i.e. When A and B are independent, then the conditional probability of A given B does not depend upon B at all.

Bayes' Theorem

Probability of A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

and probability of B given A is

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

Since $P(A \cap B) = P(B \cap A)$

$$P(A|B) = \frac{P(B|A)\ P(A)}{P(B)}$$

Let's assume S is composed of disjoint subsets A_i, thus $S = \cup_i A_i$
Further

$$B = B \cap S = B \cap (\cup_i A_i) = \cup_i (B \cap A_i)$$

$$\begin{aligned} P(B) &= \sum_i P(B \cap A_i) \\ &= \sum_i P(B|A_i) P(A_i) \end{aligned}$$

$$P(A|B) = \frac{P(B|A)\ P(A)}{\sum_i P(B|A_i) P(A_i)}$$

An example of Bayes' Theorem

The probability of having AIDS(prior to any tests) for a given population

$P(\text{AIDS}) = 0.001$
$P(\text{no AIDS}) = 0.999$

Consider a test with results + or − and conditional probabilities as

$P(+|\text{AIDS}) = 0.98$
$P(-|\text{AIDS}) = 0.02$

$P(+|\text{no AIDS}) = 0.03$
$P(-|\text{no AIDS}) = 0.98$

Follwing questions can be asked and answered:

- What is the probability for find $+(-)$ result for the population?
  $0.03095\ (0.96905)$
- If the result is $+$, what is the likelihood of person suffring from AIDS?
  0.03166
- If the result is $-$, what is the likelihood of person still suffring from AIDS?
  2.06×10^{-5}