CHAPTER 1

Completeness of $\mathbb{R}$

1.1. Completeness

$\mathbb{R}$ is an ordered Archimedean field - so is $\mathbb{Q}$. What makes $\mathbb{R}$ special is that it is **complete**. To understand this notion, we first need a couple of definitions :

DEFINITION 1.1.1. Given an ordered set X and $A \subset X$, an element $x \in X$ is called an **upper bound** of A if $x \geq a$, $\forall a \in A$.

A special kind of upper bound is

DEFINITION 1.1.2. $s \in X$ is called the **least upper bound** of A, denoted by l.u.b. A or $\sup A$ if

(i) s is an upper bound of A.

(ii) if $x \in X$ is an upper bound of A, then $x \geq s$.

LEMMA 1.1.3. *The least upper bound of a set A, if it exists, is unique.*

PROOF. Let s_1 and s_2 be two least upper bounds of A. Now since s_1 is an upper bound of A (by (i) of definition 1.1.1) and s_2 is a least upper bound, (ii) of definition 1.1.1 shows that $s_1 \geq s_2$. Similarly since s_2 is an upper bound of A and s_1 is a least upper bound, we have $s_2 \geq s_1$. Thus $s_1 = s_2$. ☐

AXIOM 1.1.4. ***Axiom of Completeness*** *If $A \subset \mathbb{R}$ has an upper bound, then it has a least upper bound (*$\sup A$ *may or may not be an element of A).*

PROBLEM 1.1.5. Prove that the bounded subset $S \subset \mathbb{Q} = \left\{r \in \mathbb{Q} : r^2 < 2\right\}$ has no least upper bound in $\mathbb{Q}$. (Hence $\mathbb{Q}$ is not complete!)

1.2. Density of $\mathbb{Q}$ in $\mathbb{R}$

THEOREM 1.2.1. ***(Archimedean property)*** *(i) Given any number $x \in \mathbb{R}$, $\exists n \in \mathbb{N}$ satisfying $n > x$.*

(ii) Given any real number $y > 0$, $\exists n \in \mathbb{N}$ satisfying $\frac{1}{n} < y$.

PROOF. (i) says that $\mathbb{N}$ is not bounded above. Assume to the contrary that it is. Then $\alpha = \sup \mathbb{N}$ will exist. Since $\alpha - 1$ is not an upper bound of $\mathbb{N}$, there will be $n \in \mathbb{N} : \alpha - 1 < n$. Then $\alpha < n + 1$. Since $n + 1 \in \mathbb{N}$ this contradicts the fact that α is an upper bound.

(ii) follows from (i) by letting $x = 1/y$. ☐

The above property of $\mathbb{N}$ is so well known as to seem trivial. However, this leads to a very important fact about how $\mathbb{Q}$ fits inside $\mathbb{R}$.

THEOREM 1.2.2. *For every two real numbers a and b with $a < b$, there exists a rational number r satisfying $a < r < b$.*

PROOF. Let us assume $0 \leq a < b$ - the other cases follow quickly from this one. We seek $m, n \in \mathbb{N}$:

$$a < \frac{m}{n} < b$$

From theorem 1.2.1(ii) we can choose $n \in \mathbb{N}$:

$$\frac{1}{n} < b - a.$$

With this choice of n, we now choose m so that

$$m - 1 \leq na < m.$$

This means that $a < \frac{m}{n}$. It also means that $a \geq \frac{m}{n} - \frac{1}{n}$ so that $b = (b - a) + a > \frac{1}{n} + \left(\frac{m}{n} - \frac{1}{n}\right) = \frac{m}{n}$. ☐

This result is often expressed by saying that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Corollary 1.2.3. *Given any two real numbers $a < b$, there exists an irrational number t satisfying $a < t < b$.*

Proof. Use theorem 1.2.2 with a, b replaced by $a - \sqrt{2}, b - \sqrt{2}$. □

Thus the irrationals are also dense in $\mathbb{R}$. In general, we define a set $G \subseteq \mathbb{R}$ to be dense in $\mathbb{R}$ if, given any two real numbers a and b it is possible to find $x \in G$ such that $a < x < b$.

1.3. Nested Interval Property (NIP)

Theorem 1.3.1. *If $I_1, I_2, \ldots, I_n, \ldots$ be a collection of nested closed intervals, $I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n \supseteq \ldots$ then*

$$\cap_{n=1}^{\infty} I_n \neq \emptyset$$

Proof. Denote I_n by $[a_n, b_n]$. Then $I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n \supseteq \ldots$ implies

$$a_1 \leq a_2 \leq \ldots \leq a_n \leq \ldots \leq b_n \leq \ldots \leq b_2 \leq b_1.$$

Consider the set $A = \{a_1, a_2, \ldots, a_n, \ldots\}$. Each of the b_i is an upper bound of A. Since $A \subseteq R$ has an upper bound, it has a least upper bound, say s. Now, since $s = \sup A$, it is an upper bound of a, $s \geq a_n$. Again, b_n is an upper bound of A and s is the least upper bound. Thus $b_n \geq s$. So, $a_n \leq s \leq b_n$and thus $s \in I_n$, $\forall n$. Thus $s \in \cap_{n=1}^{\infty} I_n$. □

Corollary 1.3.2. *If in the above diam $I_n \equiv b_n - a_n \to 0$, the intersection is the singleton set $\{s\}$.*

1.4. Sequence and subsequence

Definition 1.4.1. A sequence in $\mathbb{R}$ is a function f whose domain is the set of natural numbers $f : \mathbb{N} \to \mathbb{R}$.

We often denote a sequence by $a_1, a_2, \ldots$ or (a_n) where $a_n = f(n)$, $\forall n \in \mathbb{N}$. A very important notion in the theory of sequences is that of convergence :

Definition 1.4.2. A sequence (a_n) is defined to converge to a limit a, written as

$$\lim_{n\to\infty} a_n = a \qquad \text{or} \qquad (a_n) \to a$$

if $\forall \epsilon > 0, \exists N \in \mathbb{N} : n > N \implies |a_n - a| < \epsilon$.

Exercise 1.4.3. Prove that the limit of a sequence, if it exists, is unique.

Definition 1.4.4. If a sequence does not converge, it is said to diverge.

Example 1.4.5. The sequence $\left(\frac{1}{n}\right) = 1, \frac{1}{2}, \frac{1}{3}, \ldots$ converges to 0, while the sequence $\left((-1)^{n-1}\right) = 1, -1, 1, -1, \ldots$ diverges.

Definition 1.4.6. Let (a_n) be a sequence of real numbers and $n_1 < n_2 < \ldots$ be an increasing sequence of natural numbers. Then $a_{n_1}, a_{n_2}, \ldots$ is called a subsequence of (a_n) and is denoted by (a_{n_k}).

Theorem 1.4.7. *If a sequence converges then every subsequence converges to the same limit.*

Proof. Let $(a_n) \to a$. This means that $\forall \epsilon > 0$, $\exists N \in \mathbb{N} : n > N \implies |a_n - a| < \epsilon$. Since $n_k \geq k$ (why?), $k > N \implies n_k > N$ and then $|a_{n_k} - a| < \epsilon$. Thus (a_{n_k}) converges to a. □

1.5. Monotone Convergence Theorem

THEOREM 1.5.1. *If the sequence (a_n) is monotone increasing and bounded above, then it is convergent.*

PROOF. The set $\{a_n\}$ being bounded it has a least upper bound, say s, by the AoC. Since s is the least upper bound, for $\epsilon > 0$ the number $s - \epsilon$ is not an upper bound of A. This means that $\exists N \in \mathbb{N}$ such that $a_N > s - \epsilon$. Since (a_n) is increasing, $a_n \geq a_N > s - \epsilon \; \forall n > N$. Also $a_n \leq s < s + \epsilon, \forall n$. This implies that $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $n > N \implies |a_n - s| < \epsilon$. Thus (a_n) converges to s. □

COROLLARY 1.5.2. *A monotone decreasing sequence that is bounded below converges to a limit.*

We had previously proven the nested interval property using the Axiom of Completeness. However, we could also have used the monotone convergence theorem to prove it. Note that $I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n \supseteq \ldots$ implies

$$a_1 \leq a_2 \leq \ldots \leq a_n \leq \ldots \leq b_n \leq \ldots \leq b_2 \leq b_1.$$

Thus the sequence $a_1, a_2, \ldots$ is monotone increasing and bounded (each b_n is an upper bound). Then by 1.5.1 it converges to, say a. It is easy to see that this limit a is in each interval $[a_n, b_n]$ and thus, in their intersection.

Conversely, we can also use the NIP to prove MCT. If b is an upper bound of (a_n), start by considering the closed interval $I_1 = [a_1, b_1]$ where $b_1 = b$. Define $c = \frac{a_1+b_1}{2}$. If $\exists a_m > c$, define $I_2 = [a_2, b_2] = [c, b_1]$, otherwise define $I_2 = [a_2, b_2] = [a_1, c]$. Continuing similarly, we can construct a nested sequence of closed intervals $I_1 \supseteq I_2 \supseteq \ldots$ with $\operatorname{diam} I_n \to 0$. Corollary 1.3.2 implies that $\cap_{n=1}^{\infty} I_n = \{s\}$. It is not difficult to prove that $(a_n) \to s$.

1.6. Bolzano-Weierstrass Theorem

THEOREM 1.6.1. *Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

PROOF. Let M be a bound of the sequence (a_n), so that $|a_n| < M$ for all n. Consider $I_0 = [-M, M]$. We will construct a subsequence of (a_n) in the following manner :

Bisect the interval I_0 into two closed intervals $[-M, 0]$ and $[0, M]$ (note that we do not require these intervals to be disjoint - just that theit union is I_0). At least one of these is going to contain an infinite number of terms of the sequence. Call it I_1 and choose a term $a_{n_1} \in I_1$ (if both halves have an infinite number of terms of the sequence let us decide to arbitrarily choose the first one) . Next bisect I_1 similarly and name the half that contains an infinite number of terms of I_2 . Since I_2 contains an infinite number of terms of (a_n), it will be possible to choose $a_{n_2} \in I_2$ such that $n_2 > n_1$. Proceeding in this way we construct a subsequence $a_{n_1}, a_{n_2}, \ldots, a_{n_k}, \ldots$ with $n_1 < n_2 < \ldots < n_k <, \ldots$ as well as a sequence of nested closed intervals $I_1 \supseteq I_2 \supseteq \ldots \supseteq I_k \supseteq \ldots$ such that $a_{n_k} \in I_k$. By corollary 1.3.2 the intersection $\cap_{k=1}^{\infty} I_k$ contains a single element, say x. We now claim that the subsequence (a_{n_k}) converges to x.

It is easy to see that for $y, z \in I_k$ we have $|y - z| \leq \frac{M}{2^{k-1}}$. Since $a_{n_k}, x \in I_k$ it is easy to see that $\forall \epsilon > 0$, $\exists K \in \mathbb{N} : k > K \implies |a_{n_k} - x| < \epsilon$. □

In the above, we have proved the Bolzano-Weierstrass theorem by using the nested interval property. We will now outline an alternate proof using the monotone convergence theorem :

PROOF. Let (a_n) be a bounded sequence. Since $\forall m \in \mathbb{N}$ the set $A_m = \{a_n : n \geq m\}$ is bounded, we can define another sequence (b_m) by $b_m = \sup A_m$. Again, $A_{m+1} \subseteq A_m \implies b_{m+1} \leq b_m$. Thus (b_m) is a monotone decreasing bounded sequence and so by the monotone convergence theorem, it has a limit, say b.

We will now construct a subsequence of (a_n) that converges to b. Since $(b_m) \to b$ we can find $m_1 \in \mathbb{N}$ such that $|b_{m_1} - b| < \frac{1}{2}$. Since $b_{m_1} = \sup A_{m_1}$, $\exists n_1 \geq m_1 : b_{m_1} - \frac{1}{2} \leq a_{n_1} \leq b_{m_1}$. Thus

$$|a_{n_1} - b| \leq |a_{n_1} - b_{m_1}| + |b_{m_1} - b| < \frac{1}{2} + \frac{1}{2} = 1.$$

We can now find $m_2 > n_1 : |b_{m_2} - b| < \frac{1}{4}$. Again, $\exists n_2 \geq m_2 : b_{m_2} - \frac{1}{4} < a_{n_2} < b_{m_2}$ and similarly we can show that $|a_{n_2} - b| < \frac{1}{2}$. Proceeding like this, we can find $n_1 < n_2 < n_3 < \ldots |a_{n_k} - b| < \frac{1}{k} \, \forall k \in \mathbb{N}$ (use mathematical induction to prove rigorously that this can be done). This subsequence (a_{n_k}) obviously converges to b. □

1.7. Cauchy convergence

DEFINITION 1.7.1. **Cauchy sequence :** A sequence (a_n) in $\mathbb{R}$ is called a Cauchy sequence if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $m, n > N \implies$

$$|a_m - a_n| < \epsilon.$$

Note that the definition of a sequence being Cauchy, while being very similar to the definition of a sequence converging, differs from it in that it refers only to the tail of the sequence itself and not to some (usually unknown) limit.

THEOREM 1.7.2. *Every convergent sequence is a Cauchy sequence.*

PROOF. Let $(a_n) \to a$. Then $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $n > N \implies |a_m - a| < \frac{\epsilon}{2}$. Now, choose $m, n > N$. Then

$$|a_m - a_n| \leq |a_m - a| + |a_n - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

What makes the notion of a Cauchy sequence extremely important in $\mathbb{R}$ is that here the converse is also true - so that Cauchy convergence (which is easier to test) can be used as a criterion for convergence. To prove this, let us first prove the following lemma

LEMMA 1.7.3. *A Cauchy sequence is bounded.*

PROOF. Let (a_n) be a Cauchy sequence. Then $\exists m \in \mathbb{N}$ such that $n > m \implies |a_n - a_m| < 1$. Then $\forall n > m$ we have $|a_n| \leq |a_m| + |a_n - a_m| < |a_m| + 1$. Thus $|a_m| + 1$ is an upper bound for the "tail" of the sequence. This leaves out the first m terms, $a_1, a_2, \ldots, a_m$. It is easy to see that

$$M = \max \{|a_1|, |a_2|, \ldots, |a_m - 1|, |a_m| + 1\}$$

is an upper bound of (a_n). □

Now that we have a bounded sequence, we can use the Bolzano-Weierstrass theorem to prove

THEOREM 1.7.4. *Every Cauchy sequence in $\mathbb{R}$ is convergent.*

PROOF. Let (a_n) be Cauchy. Then since it is bounded, the Bolzano-Weierstrass theorem says that it must have a convergent subsequence (a_{n_k}). Let a be the limit of this subsequence. We claim that a is also the limit of (a_n).

Since (a_n) is Cauchy, $\exists N \in \mathbb{N}$ such that $m, n > N \implies |a_m - a_n| < \frac{\epsilon}{2}$. Again, since $(a_{n_k}) \to a$, we can choose $n_k > N$ such that $|a_{n_k} - a| < \frac{\epsilon}{2}$. Now if $n > n_k$ we have

$$|a_n - a| \leq |a_n - a_{n_k}| + |a_{n_k} - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

so that $(a_n) \to a$. □

Apart from the fact that the Cauchy criterion is a convenient way of figuring out whether a given sequence is converging, theorem 1.7.4 is an alternate way of characterising completeness.

1.8. Descriptions of Completeness

In the discussion so far, we had introduced the notion that $\mathbb{R}$ is complete by using the axiom of completeness. We have seen several consequences of the AoC, which include the NIP, the MCT, BW and CC. So far the logical chain of reasoning that we have seen can be summarised by

$$\text{AoC} \begin{cases} \implies & \text{NIP} \implies \text{BW} \implies \text{CC} \\ & \Uparrow \Downarrow \quad \nearrow\!\!\!\nearrow \\ \implies & \text{MCT} \end{cases}$$

We can prove the equivalence between the AoC, the NIP and the MCT by proving

PROPOSITION 1.8.1. *The axiom of completeness can be derived from the nested interval property.*

PROOF. Let b_0 be an upper bound of the set $A \subseteq \mathbb{R}$ and let $a_0 \in \mathbb{R}$ not be an upper bound of A. Consider the closed interval $I_0 = [a_0, b_0]$. Then $I_0 \cap A \neq \emptyset$. Let $c = \frac{a_0+b_0}{2}$. If $\exists a \in A$, $a > c$ then set $a_1 = c$, $b_1 = b_0$, else set $a_1 = a_0$, $b_1 = c$. This defines the closed interval $I_1 = [a_1, b_1]$. Proceeding in this fashion, we can construct a sequence of closed nested intervals $I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots$ in which $I_n = [a_n, b_n]$ where a_n is not an upper bound of A and b_n is an upper bound of A. Then by NIP, theorem 1.3.1, and its corollary 1.3.2, $\cap_{n=1}^{\infty} I_n = \{s\}$. We will now show that s is the least upper bound of A.

Let us first show that s is an upper bound of A. Assume that $\exists a \in A$, $a > s$. Since diam $I_n \to 0$, $\exists m \in \mathbb{N}$ such that diam $I_m < a - s$. Since $s \in I_m$, $a \notin I_m$ and so $a > b_m$. This is a contradiction, since b_m is an upper bound of A.

Again, let $l < s$. We have to show that l can not be an upper bound of A. In this case, we use the fact that $\exists m \in \mathbb{N}$ such that diam $I_m < s - l$. Since $s \in I_m$, $l \notin I_m$ and so $l < a_m$. Since $a_m \in A$ l can not be an upper bound of A. □

Since we have now shown the equivalence of AoC, NIP and MCT - we now only need to show that CC implies any of them and we have proven that all five of AoC,NIP, MCT, BW and CC are equivalent!

PROPOSITION 1.8.2. *CC implies MCT.*

PROOF. Let (a_n) be a monotone increasing sequence bounded above by b. We have to show that (a_n) is a Cauchy sequence. Let us assume that it is not. Then $\exists \epsilon > 0$ such that $\forall N \in \mathbb{N}$, $\exists m, n > N : |a_m - a_n| > \epsilon$. Since (a_m) is monotone increasing, this means that $\forall n \in \mathbb{N}$, $\exists m > n : a_m - a_n > \epsilon$. So, starting from $n_0 = 1$, we see that $\exists n_1 > n_0 : a_{n_1} - a_{n_0} > \epsilon$, $\exists n_2 > n_1 : a_{n_2} - a_{n_1} > \epsilon$, and so on. In this way we can build up a sequence $n_0, n_1, n_2, \dots$ with the propert that $a_{n_{i+1}} - a_{n_i} > \epsilon$. This means that $a_{n_r} - a_{n_0} > r\epsilon$. By choosing $r > \frac{b-a_1}{\epsilon}$ (by theorem 1.2.1 this is always possible), we get

$$a_{n_r} - a_1 > b - a_1 \quad \implies \quad a_{n_r} > b$$

which contradicts the fact that (a_n) is bounded by b. So (a_n) is Cauchy and thus it converges. □

PROBLEM 1.8.3. Starting from the Bolzano-Weierstrass theorem prove the nested interval property.