MA 311 : Analysis in $\mathbb{R}^n$

Problem Set 2

Due date : 28/08/09 before 2 PM

Q 1) Prove the monotone convergence theorem from the Cauchy convergence theorem.

Q 2) Starting from the nested interval property prove the axiom of completeness.

Hint : Let $I_1 = [a, b]$ *where* $a \in A$ *and* b *is a upper bound of* A. *Consider* $c = \frac{a+b}{2}$. *If* c *is an upper bound of* A, *then take* $I_2 = [a, c]$, *else take* $I_2 = [c, b]$. *Continue bisecting in this fashion to get a sequence of nested closed intervals.*

Q 3) If d_1 and d_2 are two metrics on the same set M which of the following are also metrics on M : (i) $d_1 + d_2$, (ii) $\max\{d_1, d_2\}$, (iii) $\min\{d_1, d_2\}$? If d is a metric on M, is d^2 a metric?

Q 4) By $2^{\mathbb{N}}$ we denote the set of all sequences (or "strings") of 0s and 1s. Show that if $a = (a_n), b = (b_n) \in 2^{\mathbb{N}}$ then

$$d(a, b) = \sum_{n=1}^{\infty} 2^{-n} |a_n - b_n|$$

defines a metric on $2^{\mathbb{N}}$.

Q 5) Show that the function $d_\infty : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$d_\infty(x, y) = \max\{|x_i - y_i|, i = 1, 2, \ldots, n\}$$

is a metric on $\mathbb{R}^n$. Sketch $B_\epsilon(x, d_\infty)$ in $\mathbb{R}^2$.

Q 6) Show that $\forall \epsilon > 0$, $\exists \epsilon', \epsilon'' > 0 :$ $B_{\epsilon'}(x, d_2) \subseteq B_\epsilon(x, d_\infty) \subseteq B_{\epsilon''}(x, d_2)$. Hence prove that if $A \subseteq \mathbb{R}^n$ is open according to d_2, then it is open according to d_∞ and *vice versa*.