MA311 : Analysis in $\mathbb{R}^n$

Problem set 5

Due Date : 10/11/09 before 2PM

Q 1) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{x|y|}{\sqrt{x^2+y^2}} & (x,y) \neq 0, \\ 0 & (x,y) = 0. \end{cases}$$

Show that f is not differentiable at the origin.

Q 2) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function such that $|f(x)| < |x|^2$. Show that f is differentiable at 0.

Q 3) Two functions $f, g : \mathbb{R} \to \mathbb{R}$ are called equal up to the nth order at $a \in \mathbb{R}$ if

$$\lim_{h \to 0} \frac{f(a+h) - g(a+h)}{h^n} = 0$$

(a) Show that f is differentiable at a iff there is a function g of the form $g(x) = \alpha_0 + \alpha_1 (x-a)$ such that f and g are equal up to the first order at a.

(b) If $f'(a), f''(a), \ldots, f^{(n)}(a)$ exist, show that f and the function g defined by

$$g(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!} (x-a)^i$$

are equal up to the nth order at a.

Q 4) Find the differential for the map $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$f(x,y) = (\sin(xy), \sin(x \sin y), x^y).$$

Q 5) A function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ is bilinear if for $x, x_1, x_2 \in \mathbb{R}^n$, $y, y_1 y_2 \in \mathbb{R}^m$ and $a \in \mathbb{R}$ we have

$$\begin{aligned} f(ax, y) &= af(x,y) = f(x, ay), \\ f(x_1 + x_2, y) &= f(x_1, y) + f(x_2, y), \\ f(x, y_1 + y_2) &= f(x, y_1) + f(x, y_2). \end{aligned}$$

(a) Prove that if f is bilinear, then

$$\lim_{(h,k) \to 0} \frac{|f(h,k)|}{|(h,k)|} = 0$$

(b) Prove that $Df(a,b)(x,y) = f(a,y) + f(x,b)$.