ON THE POLYNOMIAL CONVEXITY OF THE UNION THREE TOTALLY-REAL PLANES IN $\mathbb{C}^2$

SUSHIL GORAI

Abstract. In this paper we discuss local polynomial convexity at the origin of the union of three totally-real planes in $\mathbb{C}^2$. The planes, say P_0, P_1, P_2 satisfy a mild transversality condition: $P_0 \cap P_j = \{0\}$, $j = 1, 2$, which enables us to view them in Weinstock's normal form, i.e., $P_0 = \mathbb{R}^2$ and $P_j = M(A_j) := (A_j + i\mathbb{I})\mathbb{R}^2$, $j = 1, 2$, where each A_j is a 2×2 matrix with real entries. Weinstock solved the problem completely for two totally-real planes (in fact, for pairs of transverse, maximally totally-real subspaces of $\mathbb{C}^n\, \forall n \geq 2$). Using Weinstock's ideas for simplifying the model for $P_1 \cup P_2 \cup P_3$, we provide an open condition for the local polynomial convexity at $0 \in \mathbb{C}^2$ of the union of three totally-real planes.

1. Introduction and statement of results

Let K be a compact subset of $\mathbb{C}^n$. The polynomially convex hull of K is defined by $\widehat{K} := \{z \in \mathbb{C}^n : |p(z)| \leq \sup_K |p|,\, p \in \mathbb{C}[z_1, \dots, z_n]\}$. K is said to be polynomially convex if $\widehat{K} = K$. We say that a closed subset E of $\mathbb{C}^n$ is locally polynomially convex at $p \in E$ if $E \cap \overline{\mathbb{B}(p; r)}$ is polynomially convex for some $r > 0$ (here, $\mathbb{B}(p; r)$ denotes the open ball in $\mathbb{C}^n$ with centre p and radius r). In general, it is very difficult to determine whether a given compact subset of $\mathbb{C}^n$, $n \geq 2$, is polynomially convex. Weinstock [5] studied local polynomial convexity, at $0 \in \mathbb{C}^n$, of the union of a pair of maximal totally-real subspaces of $\mathbb{C}^n$ intersecting transversely at the origin. This paper is inspired by [5]. We consider the union of three totally-real planes in $\mathbb{C}^2$ intersecting at $0 \in \mathbb{C}^2$, with a mild transversality condition (by "mild" we mean that not all the three pairs of planes need to be mutually transverse). In this setting we shall give an *open* condition that is sufficient for such a union of totally-real planes to be locally polynomially convex at $0 \in \mathbb{C}^2$. Let us, however, make a brief survey of known results in this direction and make the above setting a bit more formal.

It is easy to show that if M is a totally-real subspace of $\mathbb{C}^n$, then any compact subset of M is polynomially convex. Hence, let us now consider $P_0 \cup P_1$, where P_0 and P_1 are two transverse totally-real n-dimensional subspaces of $\mathbb{C}^n$. Applying a $\mathbb{C}$-linear change of coordinate, we can assume that $P_0 = \mathbb{R}^n$. A careful look at the second subspace under the same change of coordinate gives us $P_1 = (A + i\mathbb{I})\mathbb{R}^n$ for some $A \in \mathbb{R}^{n \times n}$ (see [5] for details). We shall call this form for the pair of totally-real subspaces as *Weinstock's normal form.* We now state Weinstock's theorem.

Result 1.1 (Weinstock)**.** *Let P_0 and P_1 be two totally-real subspaces of $\mathbb{C}^n$ of maximal dimension intersecting only at $0 \in \mathbb{C}^n$. Denote the normal form for this pair as:*

$$P_0 : \mathbb{R}^n,$$
$$P_1 : (A + i\mathbb{I})\mathbb{R}^n.$$

2000 *Mathematics Subject Classification.* Primary: 32E20, 46J10.

Key words and phrases. Polynomial convexity; totally real.

$P_0 \cup P_1$ is locally polynomially convex at the origin if and only if A has no purely imaginary eigenvalue of modulus greater than 1.

No analogue of Weinstock's theorem is known for more than two totally-real subspaces. Even in $\mathbb{C}^2$, the problem of generalizing Weinstock's characterization does not seem any simpler. The works of Pascal Thomas [3, 4] give us some sense of the difficulties involved. In [3] Thomas gave an example of a one-parameter family of triples $(P_0^\varepsilon, P_1^\varepsilon, P_2^\varepsilon)$ of totally-real planes in $\mathbb{C}^2$, intersecting at $0 \in \mathbb{C}^2$, showing that polynomial convexity of each pairwise union at the origin does not imply the polynomial convexity of the union at the origin (see Result 2.3). In fact, he showed that for the above triples $(P_0^\varepsilon, P_1^\varepsilon, P_2^\varepsilon)$, the polynomial hull of $(\cup_{j=0}^2 P_j^\varepsilon) \cap \overline{\mathbb{B}(0;r)}$, $r > 0$, contains an open set in $\mathbb{C}^2$. On the other hand, Thomas also found in [4, Proposition 10] examples of N-tuples of totally-real planes containing $0 \in \mathbb{C}^2$, for each $N \geq 2$, whose union is locally polynomially convex at the origin. In this paper we present sufficient conditions for the local polynomial convexity of the union of three totally-real planes containing $0 \in \mathbb{C}^2$.

We shall follow several of Weinstock's ideas to our problem. One of the ideas is to focus on a certain normal form for the given union of planes. By exactly the same arguments as in [5], we can find a $\mathbb{C}$-linear change of coordinate relative to which:

$$\begin{aligned} &P_0 : \mathbb{R}^2 \\ &P_j : M(A_j) = (A_j + i\mathbb{I})\mathbb{R}^2, \quad j = 1, 2, \end{aligned} \tag{1.1}$$

where $A_1, A_2 \in \mathbb{R}^{2\times 2}$. (In this paper *we shall refer to a $\mathbb{C}$-linear operator and its matrix representation relative to the standard basis of $\mathbb{C}^n$ interchangeably.*) We shall call (1.1) *Weinstock's normal form for $\{P_0, P_1, P_2\}$*. Now, it is easy to see that if (P_0, P_1, P_2) is a triple of totally-real planes with $P_0 \cap P_j = \{0\}$, $j = 1, 2$ (with one of the three being designated as P_0 in case all three planes are mutually transverse), then the matrices A_1 and A_2 associated to Weinstock's normal form for the triple (P_0, P_1, P_2) is unique. In short, every triple (P_0, P_1, P_2) of totally-real planes with $P_0 \cap P_j = \{0\}$, $j = 1, 2$, is parametrized by a pair of matrices. Let us define (for matrices A and B, we shall denote $AB - BA$ as $[A, B]$)

$$\Omega := \left\{(A_1, A_2) \in (\mathbb{R}^{2\times 2})^2 \ : \ det[A_1, A_2] \neq 0, \ \#\sigma(A_1) = 2 \text{ and } i \notin \sigma(A_j)\, \forall j \right\}.$$

It is clear that $(\mathbb{R}^{2\times 2})^2 \setminus \Omega$ has Lebesgue measure zero. In the following theorem we will study the triples of totally-real planes parametrized by Ω. Let Θ, Λ be functions from $\mathbb{R}^{2\times 2} \times \mathbb{R}^{2\times 2}$ to $\mathbb{R}$ defined by

$$\begin{aligned} \Theta(A, B) &:= detA(TrB)^2 + TrAB(TrAB - TrATrB), \\ \Lambda(A, B) &:= 4det(AB) - \tfrac{1}{4}(TrATrB)^2. \end{aligned}$$

We are now in a position to state the main theorem of this paper.

Theorem 1.2. *Let P_0, P_1, P_2 be three totally-real planes containing $0 \in \mathbb{C}^2$. Assume $P_0 \cap P_j = \{0\}$ for $j = 1, 2$. Hence, let Weinstock's normal form for $\{P_0, P_1, P_2\}$ be*

$$\begin{aligned} &P_0 : \mathbb{R}^2 \\ &P_j : M(A_j) = (A_j + i\mathbb{I})\mathbb{R}^2, \quad j = 1, 2, \end{aligned}$$

and assume (A_1, A_2) belongs to parameter domain Ω. Assume further that the pairwise unions of P_0, P_1, P_2 are locally polynomially convex at $0 \in \mathbb{C}^2$. Given $j \in \{1, 2\}$, let j^{C} denote the other element in $\{1, 2\}$. Then:

(*i*) *Let* $\sigma(A_j) \subset \mathbb{R}$, $j = 1, 2$. *If*

either $detA_j det[A_1, A_2] > 0$, $j = 1, 2$,

or $detA_j det[A_1, A_2] < 0$ *and* $(detA_j)\Theta(A_j, A_{j^{\complement}}) < 0$ *for some* $j \in \{1, 2\}$,

then $P_0 \cup P_1 \cup P_2$ *is locally polynomially convex at* $0 \in \mathbb{C}^2$.

(*ii*) *Suppose* $\sigma(A_1) \subset \mathbb{R}$ *and* $\sigma(A_2) \subset \mathbb{C} \setminus \mathbb{R}$. *If*

$$\text{either } detA_1 det[A_1, A_2] < 0 \quad \text{and} \quad (detA_1)\Theta(A_1, A_2) < 0$$
$$\text{or } \Theta(A_1, A_2) < \Lambda(A_1, A_2),$$

then $P_0 \cup P_1 \cup P_2$ *is locally polynomially convex at* $0 \in \mathbb{C}^2$.

(*iii*) *Suppose* $\sigma(A_j) \subset \mathbb{C} \setminus \mathbb{R}$, $j = 1, 2$. *If* $\Theta(A_j, A_{j^{\complement}}) < \Lambda(A_1, A_2)$ *for some* $j \in \{1, 2\}$, *then* $P_0 \cup P_1 \cup P_2$ *is locally polynomially convex at* $0 \in \mathbb{C}^2$.

The above conditions are optimal in the sense that, writing $\Omega^* \subsetneq \Omega$ *to be set of pairs* $(A_1, A_2) \in \Omega$ *that satisfy the conditions in* (*i*) *or* (*ii*) *or* (*iii*) *above, there is a one-parameter family of triples* $(P_0^\varepsilon, P_1^\varepsilon, P_2^\varepsilon)$ *parametrized by* $(A_1^\varepsilon, A_2^\varepsilon) \in \Omega \setminus \Omega^*$ *such that*

- *pairwise unions of* $P_0^\varepsilon, P_1^\varepsilon, P_2^\varepsilon$ *are locally polynomially convex at the origin;*
- *the union of the above planes is not locally polynomially convex at* $0 \in \mathbb{C}^2$*; and*
- $(A_1^\varepsilon, A_2^\varepsilon) \to \partial\Omega^*$ *(considered as a subset of* Ω*) as* $\varepsilon \searrow 0$.

Lest the hypotheses of Theorem 1.2 make it seem very technical, we present a corollary to the above theorem that has appealing, concise hypothesis.

Corollary 1.3. *Let* P_0, P_1, P_2 *be three totally-real planes containing* $0 \in \mathbb{C}^2$. *Assume* $P_0 \cap P_j = \{0\}$ *for* $j = 1, 2$. *Hence, let Weinstock's normal form for* $\{P_0, P_1, P_2\}$ *be*

$$P_0 : \mathbb{R}^2$$
$$P_j : M(A_j) = (A_j + i\mathbb{I})\mathbb{R}^2, \quad j = 1, 2,$$

where $A_j \in \mathbb{R}^{2\times 2}$. *Let the pairwise unions of* P_0, P_1, P_2 *be locally polynomially convex at* $0 \in \mathbb{C}^2$. *Then* $P_0 \cup P_1 \cup P_2$ *is locally polynomially convex at* $0 \in \mathbb{C}^2$ *if either one of the following conditions holds:*

(*i*) $det[A_1, A_2] > 0$ *and* $detA_j > 0$, $j = 1, 2$,

(*ii*) $det[A_1, A_2] < 0$ *and* $detA_j < 0$, $j = 1, 2$.

The above result turns out to be a special case of Theorem 1.2 because the conditions in Corollary 1.3 imply that A_1, A_2 must have real eigenvalues. We present the details in Section 4.

The reader might wonder what happens when we drop the condition of $det[A_1, A_2] \neq 0$ in Corollary 1.3. When $det[A_1, A_2] = 0$, we can tell *precisely* when $P_0 \cup P_1 \cup P_2$ is locally polynomially convex at $0 \in \mathbb{C}^2$. However, this situation is non-generic in the space of triples of totally-real planes in $\mathbb{C}^2$. Hence, we shall address this issue — indeed, we can handle a certain non-generic family of N-tuples of totally-real planes, $N \geq 3$ — in a forthcoming work.

2. Technical preliminaries

We shall require some preliminaries to set the stage for proving the theorems. First, we state a lemma from Weinstock's paper [5] — whose proof is quite easy — that allows us to conjugate the matrices coming from Weinstock's normal form by real nonsingular matrices.

Lemma 2.1. *Let T be an invertible linear operator on $\mathbb{C}^n$ whose matrix representation with respect to the standard basis is an $n \times n$ matrix with real entries. Then T maps $M(A) \cup \mathbb{R}^n$ onto $M(TAT^{-1}) \cup \mathbb{R}^n$.*

Next, we state a lemma by Kallin [1] (also see [2]), which we shall use repeatedly. It deals with the polynomial convexity of the union of two polynomially convex sets.

Lemma 2.2 (Kallin)**.** *Let K_1 and K_2 be two compact polynomially convex subsets in $\mathbb{C}^n$. Suppose L_1 and L_2 are two compact polynomially convex subsets of $\mathbb{C}$ with $L_1 \cap L_2 = \{0\}$. Suppose further that there exists a holomorphic polynomial P satisfying the following conditions:*

(i) $P(K_1) \subset L_1$ and $P(K_2) \subset L_2$; and
(ii) $P^{-1}\{0\} \cap (K_1 \cup K_2)$ is polynomially convex.

Then $K_1 \cup K_2$ is polynomially convex.

Let us now state a result by Thomas [3] which will play the key role in our argument in the proof of the concluding part of Theorem 1.2.

Result 2.3 (Thomas, [3])**.** *There exist three pairwise transversal totally-real planes P_j, $0 \le j \le 2$, in $\mathbb{C}^2$ passing through origin such that:*

(i) $P_j \cup P_k$ is locally polynomially convex at $0 \in \mathbb{C}^2$ for all $j \neq k$;
(ii) $\widehat{((P_0 \cup P_1 \cup P_2) \cap \overline{\mathbb{B}(0;1)})}$ contains an open ball in $\mathbb{C}^2$.

Note that, in the statement (ii) of the above theorem, the radius of the closed ball has no significant role. Since the set $P_0 \cup P_1 \cup P_2$ is invariant under all real dilations, (ii) would hold true with any $\mathbb{B}(0;r)$, $r > 0$, replacing the unit ball. We will see some more discussions on these planes [3] in Section 3.

We now prove a couple of lemmas that will be used in the proof of Theorem 1.2. Both the lemmas are linear algebraic in nature. We start by stating a well-known result from linear algebra.

Lemma 2.4. *Let $A \in \mathbb{R}^{2\times 2}$ and suppose A has non-real eigenvalues $p \pm iq$. Then, there exists $S \in GL(2,\mathbb{R})$ such that*

$$S^{-1}AS = \begin{pmatrix} p & -q \\ q & p \end{pmatrix}.$$

Lemma 2.5. *Let $A_1, A_2 \in \mathbb{R}^{2\times 2}$. Suppose A_1 has two distinct eigenvalues. Then $\exists T \in GL(2,\mathbb{R})$ such that:*

(i) If A_1 has real eigenvalues and $det[A_1, A_2] \neq 0$, then

$$TA_1T^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \text{ and } TA_2T^{-1} = \begin{pmatrix} s_{21} & t_2 \\ t_2 & s_{22} \end{pmatrix} \text{ or } \begin{pmatrix} s_{21} & -t_2 \\ t_2 & s_{22} \end{pmatrix}$$

for $\lambda_j, s_{2j}, t_2 \in \mathbb{R}$, $j = 1, 2$,

(ii) If A_1 has non-real eigenvalues, then

$$TA_1T^{-1} = \begin{pmatrix} s_1 & -t_1 \\ t_1 & s_1 \end{pmatrix} \text{ and } TA_2T^{-1} = \begin{pmatrix} s_{21} & -t_2 \\ t_2 & s_{22} \end{pmatrix}$$

for $s_j, s_{2j}, t_j \in \mathbb{R}$, $j = 1, 2$.

Proof. (i) Since A_1 has two distinct real eigenvalues, A_1 is diagonalizable over $\mathbb{R}$, i.e. there exists a $S \in GL(2,\mathbb{R})$ such that

$$SA_1S^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \ \lambda_1 \neq \lambda_2 \in \mathbb{R}.$$

Hence, without loss of generality, we can assume that $A_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. Suppose $A_2 = \begin{pmatrix} s_{21} & t_1 \\ t_2 & s_{22} \end{pmatrix}$, $t_j, s_{2j} \in \mathbb{R}$ for $j = 1, 2$. Observe that, $t_1 t_2 = 0 \Leftrightarrow det[A_1, A_2] = 0$. Hence, neither t_1 nor t_2 is zero. We have, since A_1 commutes with all diagonal matrices, that $GA_1G^{-1} = A_1$ for $G = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$, where $g_1 g_2 \neq 0$. We also have, after conjugating A_2 by G, that

$$GA_2G^{-1} = \begin{pmatrix} s_{21} & t_1 g_1/g_2 \\ t_2 g_2/g_1 & s_{22} \end{pmatrix}. \tag{2.1}$$

Now observe that if t_1 and t_2 are of same sign, then there exist $g_1, g_2 \in \mathbb{R} \setminus \{0\}$ such that $t_1 g_1^2 = t_2 g_2^2$. Therefore, in this case, $\widetilde{t_2} := t_1 g_1/g_2 = t_2 g_2/g_1$, and we conclude from (2.1) that $GA_2G^{-1} = \begin{pmatrix} s_{21} & \widetilde{t_2} \\ \widetilde{t_2} & s_{22} \end{pmatrix}$. If t_1 and t_2 are of different sign, then there exist $g_1, g_2 \in \mathbb{R} \setminus \{0\}$ such that $t_1 g_1^2 + t_2 g_2^2 = 0$. Therefore, in this case, the above argument, but with $\widetilde{t_2} := -t_1 g_1/g_2 = t_2 g_2/g_1$, gives the desired conclusion.

(ii) Since A_1 has non-real eigenvalues, say $s_1 \pm it_1$, in view of Lemma 2.4, we may assume that $A_1 = \begin{pmatrix} s_1 & -t_1 \\ t_1 & s_1 \end{pmatrix}$. Let $A_2 = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$, $m_j \in \mathbb{R}$, $j = 1, 2, 3, 4$, with $m_2 + m_3 \neq 0$; otherwise, there is nothing to prove.

We observe that A_1 commutes with all the matrices having the same structure as that of itself. Let $G := \begin{pmatrix} g_1 & -g_2 \\ g_2 & g_1 \end{pmatrix}$ with $g_1, g_2 \in \mathbb{R}$, $g_1^2 + g_2^2 = 1$ and $g_1 g_2 \neq 0$. Therefore, $GA_1G^{-1} = A_1$, and

$$\begin{aligned} GA_2G^{-1} &= \begin{pmatrix} g_1 & -g_2 \\ g_2 & g_1 \end{pmatrix} \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \begin{pmatrix} g_1 & g_2 \\ -g_2 & g_1 \end{pmatrix} \\ &= \begin{pmatrix} g_1^2 m_1 - g_1 g_2 (m_2 + m_3) + g_2^2 m_4 & g_1^2 m_2 + g_1 g_2 (m_1 - m_4) - g_2^2 m_3 \\ g_1^2 m_3 + g_1 g_2 (m_1 - m_4) - g_2^2 m_2 & g_2^2 m_1 + g_1 g_2 (m_2 + m_3) + g_1^2 m_4 \end{pmatrix} \\ &=: \begin{pmatrix} f_1(g_1, g_2) & f_2(g_1, g_2) \\ f_3(g_1, g_2) & f_4(g_1, g_2) \end{pmatrix}. \end{aligned}$$

Viewing $f_2(g_1, g_2) + f_3(g_1, g_2) = 0$ as a quadratic equation in g_1, g_2 we get:

$$(m_2 + m_3)(g_1^2 - g_2^2) + 2(m_1 - m_4) g_1 g_2 = 0.$$

Since $m_2 + m_3 \neq 0$ and $g_1 g_2 \neq 0$, the above is equivalent to the following quadratic equation in $\mu := g_1/g_2$:

$$\mu^2 + 2\frac{m_1 - m_4}{m_2 + m_3}\mu - 1 = 0. \tag{2.2}$$

The discriminant of the above quadratic is $4\left(\frac{m_1 - m_4}{m_2 + m_3}\right)^2 + 4$, which is greater than zero for all $m_j \in \mathbb{R}$, $j = 1, 2, 3, 4$. Hence (2.2) has a real root, say μ_1. Therefore, taking $G = \begin{pmatrix} \mu_1 & -1 \\ 1 & \mu_1 \end{pmatrix}$, we have the desired conclusion. □

3. The proof of Theorem 1.2

The proof of Theorem 1.2: Throughout this proof our computations will involve conjugates of A_1, A_2 by an appropriate $T \in GL(2, \mathbb{R})$. Also, for simplicity of notation, we shall denote the planes $T(P_j)$ as P_j, $j = 0, 1, 2$.

(i) The treatment of this part falls into two cases depending on the hypothesis.

Case I. *When $detA_j det[A_1, A_2] > 0$, $j = 1, 2$.*

We now divide the proof of this case into two subcases depending on the sign of $det[A_1, A_2]$.

Sub-case (*a*) *When $det[A_1, A_2] > 0$.*

Note that, since $det[A_1, A_2] > 0$, we have $detA_j > 0$, $j = 1, 2$. First, we shall prove two claims to reduce the pair of matrices (A_1, A_2) into simpler form, which will enable us to use Kallin's lemma.

Claim 1. *If $det[A_1, A_2] > 0$, then each A_j has distinct eigenvalues.*

Proof of Claim 1. Suppose A_1 does not have distinct eigenvalues. Let $\sigma(A_1) = \{\lambda_1\}$. As $A_1 \in \mathbb{R}^{2\times 2}$, $\lambda_1 \in \mathbb{R}$. Thus, there exists $T \in GL(2, \mathbb{R})$ such that

$$TA_1T^{-1} = \begin{pmatrix} \lambda_1 & \mu \\ 0 & \lambda_1 \end{pmatrix}.$$

Let us write $TA_2T^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then, by a simple computation, we see that

$$det[TA_1T^{-1}, TA_2T^{-1}] = -c^2\mu^2 \leq 0,$$

which is a contradiction.

Claim 2. *If $det[A_1, A_2] > 0$, then there exists a $T \in GL(2, \mathbb{R})$ such that*

$$TA_1T^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},\ TA_2T^{-1} = \begin{pmatrix} s_{21} & t_2 \\ t_2 & s_{22} \end{pmatrix}.$$

Proof of Claim 2. In view of Claim 1, we conclude that the eigenvalues of A_1 are real and distinct. Hence, applying Part (i) of Lemma 2.5, we get that there exists $T \in GL(2, \mathbb{R})$ such that

$$TA_1T^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \text{ and } TA_2T^{-1} = \begin{pmatrix} s_{21} & t_2 \\ t_2 & s_{22} \end{pmatrix} \text{ or } \begin{pmatrix} s_{21} & -t_2 \\ t_2 & s_{22} \end{pmatrix}.$$

Suppose $TA_2T^{-1} = \begin{pmatrix} s_{21} & -t_2 \\ t_2 & s_{22} \end{pmatrix}$. Again calculating the commutator, we note that

$$det[TA_1T^{-1}, TA_2T^{-1}] = -t_2^2(\lambda_2 - \lambda_1)^2 \leq 0,$$

which is a contradiction. Hence the claim.

We can now resume the proof of Part (i) of Theorem 1.2. In view of Claim 2, we always get a $T \in GL(2, \mathbb{R})$ such that

$$TA_1T^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \text{ and } TA_2T^{-1} = \begin{pmatrix} s_{21} & t_2 \\ t_2 & s_{22} \end{pmatrix}.$$

Let $K_j = P_j \cap \overline{\mathbb{B}(0; 1)}$, $j = 0, 1, 2$. We shall use Kallin's lemma to show the polynomial convexity of $K_0 \cup K$, where $K := K_1 \cup K_2$. By our hypothesis on pairwise unions and by homogeneity, K is polynomially convex. Let us consider the polynomial $F(z, w) = z^2 + w^2$. Clearly,

$$F(K_0) \subset \{z \in \mathbb{C} : z \geq 0\}. \tag{3.1}$$

For $(z, w) \in K_1$, we have

$$F(z, w) = F((\lambda_1 + i)x, (\lambda_2 + i)y) = (\lambda_1^2 - 1)x^2 + (\lambda_2^2 - 1)y^2 + 2i(\lambda_1 x^2 + \lambda_2 y^2), \tag{3.2}$$

and, for $(z,w)\in K_2$,

$$\begin{aligned}F(z,w) &= F((s_{21}+i)x+t_2y, t_2x+(s_{22}+i)y)\\ &= (s_{21}^2+t_2^2-1)x^2+(s_{22}^2+t_2^2-1)y^2+2(s_{21}+s_{22})t_2xy\\ &\qquad\qquad + 2i(s_{21}x^2+s_{22}y^2+2t_2xy). \end{aligned}\tag{3.3}$$

By hypothesis, we have

$$detA_1 = det(TA_1T^{-1})>0 \Longrightarrow \lambda_1\lambda_2>0, \tag{3.4}$$

$$detA_2 = det(TA_2T^{-1})>0 \Longrightarrow s_{21}s_{22}-t_2^2>0. \tag{3.5}$$

In view of (3.4) and (3.5), equations (3.2) and (3.3) give us $F(K)\subset(\mathbb{C}\setminus\mathbb{R})\cup\{0\}$. Therefore, in view of (3.1), we get

$$\widehat{F(K_0)}\cap\widehat{F(K)}=\{0\}, \tag{3.6}$$

and

$$F^{-1}\{0\}\cap(K_0\cup K)=\{0\}. \tag{3.7}$$

Therefore, from (3.6) and (3.7), all the conditions of Lemma 2.2 are satisfied. Hence, $K_0\cup K$ is polynomially convex.

Sub-case (*b*) *When* $det[A_1,A_2]<0$.

We note that, since $det[A_1,A_2]<0$, $detA_j<0$, $j=1,2$. Again, for this part, we need to obtain simpler form of the matrices.

Claim 3. *It suffices to work with the union* $\mathbb{R}^2\cup M(\mathcal{A}_1)\cup M(\mathcal{A}_2)$*, where:*

$$\mathcal{A}_1=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} \text{ and } \mathcal{A}_2=\begin{pmatrix}s_{21} & -t_2\\ t_2 & s_{22}\end{pmatrix}.$$

Proof of Claim 3. Since $detA_j<0$ for $j=1,2$, each A_j must have real distinct eigenvalues. Hence, in view of Lemma 2.5, we can find a $T\in GL(2,\mathbb{R})$ such that

$$TA_1T^{-1}=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} \text{ and } TA_2T^{-1}=\begin{pmatrix}s_{21} & t_2\\ t_2 & s_{22}\end{pmatrix} \text{ or } \begin{pmatrix}s_{21} & -t_2\\ t_2 & s_{22}\end{pmatrix},$$

for $\lambda_j, s_{2j}, t_2\in\mathbb{R}$, $j=1,2$. Let $\mathcal{A}_j=TA_jT^{-1}$ for $j=1,2$. With the *first* alternative for $\mathcal{A}_2$, we get $det[A_1,A_2]=det[\mathcal{A}_1,\mathcal{A}_2]=(\lambda_1-\lambda_2)^2t_2^2>0$, which is a contradiction to the assumption that $det[A_1,A_2]<0$. Hence,

$$\mathcal{A}_2=\begin{pmatrix}s_{21} & -t_2\\ t_2 & s_{22}\end{pmatrix}.$$

The claim follows from Lemma 2.1.

Write $K_j=P_j\cap\overline{\mathbb{B}(0;1)}$, $j=0,1,2$. We now show that $K_0\cup K$ is polynomially convex, where $K=K_1\cup K_2$. Consider the polynomial $F(z,w)=z^2-w^2$. Clearly,

$$F(K_0)\subset\mathbb{R}\subset\mathbb{C}. \tag{3.8}$$

For $(z,w)\in K_1$, we have

$$\begin{aligned}F(z,w) &= F((\lambda_1+i)x,(\lambda_2+i)y)\\ &= (\lambda_1^2-1)x^2+(1-\lambda_2^2)y^2+2i(\lambda_1x^2-\lambda_2y^2),\end{aligned}\tag{3.9}$$

and, for $(z,w)\in K_2$,

$$\begin{aligned}F(z,w) &= F((s_{21}+i)x - t_2y, t_2x+(s_{22}+i)y)\\ &= (s_{21}^2 - t_2^2 - 1)x^2 + (1-s_{22}^2+t_2^2)y^2 - 2(s_{21}+s_{22})t_1xy\\ &\qquad + 2i(s_{21}x^2 - s_{22}y^2 - 2t_2xy). \qquad (3.10)\end{aligned}$$

We now show that $\widehat{F(K)}\cap\widehat{F(K_0)}=\{0\}$. We have

$$detA_1 < 0 \Longrightarrow \lambda_1\lambda_2 < 0, \qquad (3.11)$$

$$detA_2 < 0 \Longrightarrow s_{21}s_{22} + t_2^2 < 0. \qquad (3.12)$$

In view of (3.11) and (3.12), expressions (3.9) and (3.10) give us $F(K)\subset(\mathbb{C}\setminus\mathbb{R})\cup\{0\}$. Hence, in view of (3.8), we have

$$\widehat{F(K)}\cap\widehat{F(K_0)}=\{0\} \text{ and } F^{-1}\{0\}\cap K=\{0\}.$$

We also have $F^{-1}\{0\}\cap K_0=\{(x,y)\in K_0 : x=\pm y\}$. Hence, $F^{-1}\{0\}\cap(K\cup K_0)$ is polynomially convex. Therefore, by Lemma 2.2, we are done.

Case II. *When $detA_j det[A_1,A_2]<0$ and $(detA_j)\Theta(A_j,A_{j^{\complement}})<0$ for some $j\in\{1,2\}$*
We have $\sigma(A_j)\subset\mathbb{R}$, $j=1,2$. The hypothesis in this case of Part (i) is symmetric in $j\in\{1,2\}$. Hence, we can assume that $j=1$. We again divide the proof into two cases depending on the sign of $det[A_1,A_2]$.

Sub-case (*a*) *When $det[A_1,A_2]>0$.*
Since $det[A_1,A_2]>0$, in view of Claim 2 in the proof of Case I above, there exists a $T\in GL(2,\mathbb{R})$ such that

$$TA_1T^{-1}=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} \text{ and } TA_2T^{-1}=\begin{pmatrix}s_{21} & t_2\\ t_2 & s_{22}\end{pmatrix}.$$

Let $K_j=P_j\cap\overline{\mathbb{B}(0;1)}$, $j=0,1,2$, and write $K:=K_1\cup K_2$. We shall show $K\cup K_0$ is polynomially convex by using Kallin's lemma. As discussed above, K is polynomially convex. Consider the polynomial $F(z,w)=z^2-w^2$. Clearly, (3.8) and (3.9) hold as K_1 and the polynomial that we have considered here is same as in Sub-case (b) of Case I above. For $(z,w)\in K_2$:

$$\begin{aligned}F(z,w) &= F((s_{21}+i)x + t_2y, t_2x+(s_{22}+i)y)\\ &= (s_{21}^2 - t_2^2 - 1)x^2 + (1+t_2^2-s_{22}^2)y^2 + 2t_2(s_{21}+s_{22})xy + 2i(s_{21}x^2 - s_{22}y^2)\end{aligned} \qquad (3.13)$$

Since $det[A_1,A_2]>0$, it follows from our hypothesis and the observation at the beginning of Case II that $detA_1<0$ and $\Theta(A_1,A_2)>0$. Note:

$$detA_1 < 0 \Longrightarrow \lambda_1\lambda_2 < 0, \qquad (3.14)$$

$$\Theta(A_1,A_2) > 0 \Longrightarrow s_{21}s_{22} < 0. \qquad (3.15)$$

Therefore, in view of (3.14) and (3.15), we have from (3.8), (3.9) and (3.13) that

$$F(K)\subset(\mathbb{C}\setminus\mathbb{R})\cup\{0\} \text{ and } F^{-1}\{0\}\cap K=\{0\}.$$

Hence,

$$\widehat{F(K)}\cap\widehat{F(K_0)}=\{0\}.$$

We also have $F^{-1}\{0\}\cap K_0=\{(x,y)\in K_0 : x=\pm y\}$. Hence, $F^{-1}\{0\}\cap(K\cup K_0)$ is polynomially convex. Therefore, all the conditions of Kallin's lemma are satisfied. Hence, $K\cup K_0$ is polynomially convex.

Sub-case (*b*) *When* $det[A_1, A_2] < 0$.

In view of $det[A_1, A_2] < 0$, we have, by applying Part (i) of Lemma 2.5, that there exists a $T \in GL(2, \mathbb{R})$ such that

$$TA_1T^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \text{ and } TA_2T^{-1} = \begin{pmatrix} s_{21} & -t_2 \\ t_2 & s_{22} \end{pmatrix}$$

(i.e., the second alternative for TA_2T^{-1} cannot occur since $det[A_1, A_2] < 0$).

As above, let $K_j = P_j \cap \overline{\mathbb{B}(0;1)}$, $j = 0, 1, 2$, and $K = K_1 \cup K_2$. This time, we consider the polynomial $F(z, w) = z^2 + w^2$. When $(z, w) \in K_1$, $F(z, w)$ is as in equation (3.2), and, for $(z, w) \in K_2$,

$$\begin{aligned} F(z, w) &= F((s_{21} + i)x - t_2y, t_2x + (s_{22} + i)y) \\ &= (s_{21}^2 + t_2^2 - 1)x^2 + (s_{22}^2 + t_2^2 - 1)y^2 + 2(s_{22} - s_{21})t_2xy + 2i(s_{21}x^2 + s_{22}y^2). \end{aligned} \tag{3.16}$$

Since $det[A_1, A_2] < 0$, our hypothesis forces $detA_1 > 0$, and

$$detA_1 > 0 \Longrightarrow \lambda_1\lambda_2 > 0. \tag{3.17}$$

Furthermore $\Theta(A_1, A_2) < 0$, and

$$\Theta(A_1, A_2) < 0 \Longrightarrow s_{21}s_{22} > 0. \tag{3.18}$$

In view of equations (3.17) and (3.18), the expressions in (3.1), (3.2) and (3.16) give us $F(K) \subset (\mathbb{C} \setminus \mathbb{R}) \cup \{0\}$, whence,

$$\widehat{F(K_0)} \cap \widehat{F(K)} = \{0\}.$$

We also have $F^{-1}\{0\} \cap (K_0 \cup K) = \{0\}$. Therefore, all the conditions of Kallin's lemma are satisfied. Hence $K_0 \cup K$ is polynomially convex.

(ii) We shall divide the proof into two cases depending on two different hypotheses.

Case I. *When* $detA_1 det[A_1, A_2] < 0$ *and* $(detA_1)\Theta(A_1, A_2) < 0$.

Since A_1 has real distinct eigenvalues, we can invoke Part (i) of Lemma 2.5 to get that there exists a $T \in GL(2, \mathbb{R})$ such that

$$TA_1T^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \text{ and } TA_2T^{-1} = \begin{pmatrix} s_{21} & t_2 \\ t_2 & s_{22} \end{pmatrix} \text{ or } \begin{pmatrix} s_{21} & -t_2 \\ t_2 & s_{22} \end{pmatrix}.$$

Since $\sigma(A_2) \subset \mathbb{C} \setminus \mathbb{R}$, we have $TA_2T^{-1} = \begin{pmatrix} s_{21} & -t_2 \\ t_2 & s_{22} \end{pmatrix}$. As before, using these special conjugacy representatives, we see that

$$det[A_1, A_2] = det[TA_1T^{-1}, TA_2T^{-1}] = -t_2^2(\lambda_1 - \lambda_2)^2 < 0.$$

Hence, we have the forms of the matrices to be the same as in Sub-case (b) of Case II of Part (i). Also, as $det[A_1, A_2] < 0$, we have $detA_1 > 0$ and $\Theta(A_1, A_2) < 0$. Hence, all the inputs are the same as for the proof of Sub-case (b) of Case II of Part (i). Hence, that proof works here too.

Case II. *When* $\Theta(A_1, A_2) < \Lambda(A_1, A_2)$.

Since $\sigma(A_2) \subset \mathbb{C} \setminus \mathbb{R}$, applying Part (ii) of Lemma 2.5, we get that $\exists T \in GL(2, \mathbb{R})$ such that

$$TA_2T^{-1} = \begin{pmatrix} s_1 & -t_1 \\ t_1 & s_1 \end{pmatrix} \text{ and } TA_1T^{-1} = \begin{pmatrix} s_{21} & -t_2 \\ t_2 & s_{22} \end{pmatrix}.$$

Write $K_j = P_j \cap \overline{\mathbb{B}(0;1)}$, $j = 0, 1, 2$. We shall again use Kallin's lemma to show the polynomial convexity of $K_0 \cup K_1 \cup K_2$. Consider the polynomial $F(z, w) = z^2 + w^2$.

When $(z,w)\in K_0$, $F(z,w)$ is as in (3.1). For $(z,w)\in K_1$, $F(z,w)$ is as in equation (3.16), and for $(z,w)\in K_2$, we have

$$\begin{aligned}F(z,w) &= F((s_1+i)x-t_1y, t_1x+(s_1+i)y)\\ &= (s_1^2+t_1^2-1)(x^2+y^2)+2is_1(x^2+y^2).\end{aligned}\tag{3.19}$$

Let $K=K_1\cup K_2$. Recall: from homogeneity of the totally-real planes and the hypothesis that the pairwise unions of the given totally-real planes are locally polynomially convex at the origin, K is polynomially convex. By hypotheses, we get that

$$\Theta(A_1,A_2)<\Lambda(A_1,A_2)\Longrightarrow s_{21}s_{22}>0 \tag{3.20}$$

Hence, in view of (3.20), (3.1), (3.19) and (3.16), we conclude that

$$F(K_0)\subset\{z\in\mathbb{C}: z\geq 0\},\quad F(K)\subset(\mathbb{C}\setminus\mathbb{R})\cup\{0\},$$

and $F^{-1}\{0\}\cap(K_0\cup K)=\{0\}$. Therefore, by Lemma 2.2, $K_0\cup K$ is polynomially convex.

(iii) In this case, we are given that $\sigma(A_k)\subset\mathbb{C}\setminus\mathbb{R}$, $k=1,2$. Suppose

$$\Theta(A_j,A_{j^{\mathsf{C}}})<\Lambda(A_1,A_2) \text{ for some } j\in\{1,2\}.$$

We may assume without loss of generality that $j=1$. Invoking Part (ii) of Lemma 2.5, we get that there exists a $T\in GL(2,\mathbb{R})$ such that

$$TA_2T^{-1}=\begin{pmatrix}s_1 & -t_1\\ t_1 & s_1\end{pmatrix} \text{ and } TA_1T^{-1}=\begin{pmatrix}s_{21} & -t_2\\ t_2 & s_{22}\end{pmatrix}.$$

Hence, the forms of the matrices are same as in Case II of the proof of Part (ii). We also note that the hypothesis in (ii) rendered in terms of s_1, s_{21}, s_{22}, t_1 and t_2 is same as that of Case II of Part (ii). Hence, the same proof is valid.

We now address the concluding part of this theorem. For $\varepsilon>0$ small enough, consider the following planes:

$$\begin{aligned}P_0^\varepsilon &: \mathbb{R}^2\\ P_j^\varepsilon &: (A_j^\varepsilon+i\mathbb{I})\mathbb{R}^2,\ j=1,2,\end{aligned}$$

where $A_j^\varepsilon\in\mathbb{R}^{2\times2}$ have the following form:

$$A_1^\varepsilon=\begin{pmatrix}\frac{\varepsilon}{\sqrt{3}(1+\varepsilon)} & \frac{1}{1+\varepsilon}\\ -\frac{1}{1-\varepsilon} & -\frac{\varepsilon}{\sqrt{3}(1-\varepsilon)}\end{pmatrix} \text{ and } A_2^\varepsilon=\begin{pmatrix}-\frac{\varepsilon}{\sqrt{3}(1+\varepsilon)} & \frac{1}{1+\varepsilon}\\ -\frac{1}{1-\varepsilon} & \frac{\varepsilon}{\sqrt{3}(1-\varepsilon)}\end{pmatrix}. \tag{3.21}$$

The above triples are obtained by applying the $\mathbb{C}$-linear change of coordinate $(z,w)\longmapsto(z+w, i(w-z))$ to a class of triples of totally-real graphs studied by P. Thomas [3] in proving Result 2.3. Hence, the pairwise unions of the above totally-real planes are locally polynomially convex at 0 but their union is not locally polynomially convex at 0. We now show that show that $(A_1^\varepsilon,A_2^\varepsilon)\in\Omega\setminus\Omega^*$ and $(A_1^\varepsilon,A_2^\varepsilon)\to\partial\Omega^*$ as $\varepsilon\to0$.

An elementary computation gives:

$$\begin{aligned}\sigma(A_1^\varepsilon) &= \left\{\frac{-\varepsilon^2+\sqrt{4\varepsilon^2-3}}{\sqrt{3}(1-\varepsilon^2)}, \frac{-\varepsilon^2-\sqrt{4\varepsilon^2-3}}{\sqrt{3}(1-\varepsilon^2)}\right\},\\ \sigma(A_2^\varepsilon) &= \left\{\frac{\varepsilon^2+\sqrt{4\varepsilon^2-3}}{\sqrt{3}(1-\varepsilon^2)}, \frac{\varepsilon^2-\sqrt{4\varepsilon^2-3}}{\sqrt{3}(1-\varepsilon^2)}\right\}.\end{aligned}$$

Clearly, for $\varepsilon\ :\ 0<\varepsilon\ll 1$, $(A_1^\varepsilon, A_2^\varepsilon)\in\Omega$. Now, from (iii) (read in the contrapositive) in the statement of Theorem 1.2 and Result 2.3, we already know that $(A_1^\varepsilon, A_2^\varepsilon)\notin\Omega^*$ $\forall\varepsilon\ :\ 0<\varepsilon\ll 1$. It is easy to see that $det[A_1^\varepsilon, A_2^\varepsilon]\neq 0\ \forall\varepsilon>0$. Hence,

$$(A_1^\varepsilon, A_2^\varepsilon)\in\Omega\setminus\Omega^*\ \forall\varepsilon\ :\ 0<\varepsilon\ll 1.$$

Now observe that:

$$\lim_{\varepsilon\to 0} A_j^\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} =: A_j^0,\ j=1,2.$$

Now consider the family $(B_1^\varepsilon, B_2^\varepsilon)\in\Omega^*$: $B_1^\varepsilon=\begin{pmatrix} \varepsilon & -1 \\ 1 & \varepsilon \end{pmatrix}, B_2^\varepsilon=\begin{pmatrix} \varepsilon & -1 \\ 1 & 2\varepsilon \end{pmatrix}$, $\varepsilon>0$. Here, $(B_1^\varepsilon, B_2^\varepsilon)$ satisfies the conditions in (iii); note that $\Lambda(B_1^\varepsilon, B_2^\varepsilon)-\Theta(B_1^\varepsilon, B_2^\varepsilon)=2\varepsilon^2>0$. Also, $\lim_{\varepsilon\to 0} B_j^\varepsilon = A_j^0$, $j=1,2$. Therefore, $(A_1^0, A_2^0)\in\partial\Omega^*$. □

4. The proof of Corollary 1.3

For the proof of Corollary 1.3, we first need the following claim:

Claim. *If $det[A_1, A_2]>0$, then A_j cannot have non-real eigenvalues, $j=1,2$.*

Proof of Claim. Suppose A_1 has non-real eigenvalues. Then, with T as in Lemma 2.5, it follows that

$$det[TA_1T^{-1}, TA_2T^{-1}]=-t_1^2(s_{22}-s_{21})^2\leq 0.$$

This is a contradiction, whence the claim.

Also note that in case $detA_j<0$, A_j must have distinct real eigenvalues, $j=1,2$. We see that, under either hypothesis, we can invoke Part (i) of Theorem 1.2. Hence the result follows.

Acknowledgements. I would like to thank Gautam Bharali for many useful comments on this paper. I would also like to thank the referee for comments on improving the exposition in this paper.

References

[1] E. Kallin, *Fat polynomially convex sets*, Function Algebras, Proc. Internat. Sympos. on Function Algebras, Tulane University, 1965, Scott Foresman, Chicago, IL, 1966, pp. 149-152.

[2] P. J. de Paepe, *Eva Kallin's lemma on polynomial convexity*, Bull. London Math. Soc. **33** (2001), no. 1, 1-10.

[3] Pascal J. Thomas, *Unions minimales de n-plans réels d'enveloppe égale à* $\mathbb{C}^n$, Several Complex Variables and Complex Geometry, Part 1 (Santa Cruz, CA, 1989), 233-244, Proc. Sympos. Pure Math., **52**, Part 1, Amer. Math. Soc., Providence, RI, 1991.

[4] Pascal J. Thomas, *Enveloppes polynomiales d'unions de plans réels dans* $\mathbb{C}^n$, Ann. Inst. Fourier (Grenoble) **40** (1990), no. 2, 371-390.

[5] B.M. Weinstock, *On the polynomial convexity of the union of two totally-real subspaces of* $\mathbb{C}^n$, Math. Ann. **282** (1988), no. 1, 131-138.

Statistics and Mathematics Unit, Indian Statistical Institute Bangalore, R. V. C. E. Post, Bangalore – 560 059

E-mail address: sushil.gorai@gmail.com