ON POLYNOMIAL CONVEXITY OF COMPACT SUBSETS OF TOTALLY-REAL SUBMANIFOLDS IN $\mathbb{C}^n$

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Abstract. Let K be a compact subset of a totally-real manifold M, where M is either a $\mathcal{C}^2$-smooth graph in $\mathbb{C}^{2n}$ over $\mathbb{C}^n$, or $M = u^{-1}\{0\}$ for a $\mathcal{C}^2$-smooth submersion u from $\mathbb{C}^n$ to $\mathbb{R}^{2n-k}$, $k \leq n$. In this case we show that K is polynomially convex if and only if for a fixed neighbourhood U, defined in terms of the defining functions of M, there exists a plurisubharmonic function Ψ on $\mathbb{C}^n$ such that $K \subset \{\Psi < 0\} \subset U$.

1. Introduction and statements of the results

The *polynomially convex hull* of a compact subset K of $\mathbb{C}^n$ is defined as $\widehat{K} := \{z \in \mathbb{C}^n : |p(z)| \leq \sup_K |p|, p \in \mathbb{C}[z_1, \dots, z_n]\}$. We say that K is *polynomially convex* if $\widehat{K} = K$. As a motivation for studying polynomial convexity we discuss briefly some of its connections with the theory of uniform approximation by polynomials. Let $\mathcal{P}(K)$ denote the uniform algebra on K generated by holomorphic polynomials. A fundamental question in the theory of uniform algebras is to characterize the compacts K of $\mathbb{C}^n$ for which

$$\mathcal{P}(K) = \mathcal{C}(K), \tag{1.1}$$

where $\mathcal{C}(K)$ is the class of all continuous functions on K. For $K \subset \mathbb{R} \subset \mathbb{C}$, (1.1) follows from Stone-Weierstrass theorem. More generally, Lavrentiev [12] showed that $K \subset \mathbb{C}$ *has Property* (1.1) *if and only if K is polynomially convex and has empty interior.* In contrast, no such characterization is available for compact subsets of $\mathbb{C}^n$, $n \geq 2$. Since the maximal ideal space of $\mathcal{P}(K)$, $K \subset \mathbb{C}^n$, is identified with $\widehat{K}$ via Gelfand's theory of commutative Banach algebras (see [5] for details), we observe that

$$\mathcal{P}(K) = \mathcal{C}(K) \Longrightarrow \widehat{K} = K.$$

With the assumption that K is polynomial convex, there are several results, for instance see [1, 2, 16, 20, 22], that describe situations when (1.1) holds. Unless there is some way to determine whether $K \subset \mathbb{C}^n$, $n \geq 2$, is polynomially convex—which, in general, is very difficult to determine—all of these results are somewhat abstract. One such result is due to O'Farrell, Preskenis and Walsh [16] which, in essence, says that polynomial convexity is sufficient for certain classes of compact subsets of $\mathbb{C}^n$ to satisfy Property (1.1). More precisely:

Result 1.1 (O'Farrell, Preskenis and Walsh)**.** *Let K be a compact polynomially convex subset of $\mathbb{C}^n$. Assume that E is a closed subset of K such that $K \setminus E$ is locally contained in totally-real manifold. Then*

$$\mathcal{P}(K) = \{f \in \mathcal{C}(K) : f|_E \in \mathcal{P}(E)\}.$$

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By Result 1.1, if K is a compact polynomially convex subset of a totally-real submanifold of $\mathbb{C}^n$, then $\mathcal{P}(K) = \mathcal{C}(K)$. In view of this fact, one is motivated to focus—with the goal of polynomial convexity—on characterizing the class of compact subsets of $\mathbb{C}^n$ that lie locally in some totally-real submanifold of $\mathbb{C}^n$.

A totally-real set M of $\mathbb{C}^n$ is locally polynomially convex at each $p \in M$, i.e., for each point $p \in M$ there exists a ball $B(p,r)$ in $\mathbb{C}^n$ such that $M \cap \overline{B(p,r)}$ is polynomially convex (see [21] for a proof in $\mathbb{C}^2$ and [10, 9] for a proof in $\mathbb{C}^n$, $n \geq 2$). In general, an arbitrary compact subset of a totally-real submanifold in $\mathbb{C}^n$ is not necessarily polynomially convex, as shown by the following example due to Wermer [10, Example 6.1]: let

$$M := \{(z, f(z)) \in \mathbb{C}^2 : z \in \mathbb{C}\},$$

where

$$f(z) = -(1+i)\overline{z} + iz\overline{z}^2 + z^2\overline{z}^3.$$

It is easy to see that M is totally-real. Consider the compact subset $K := \{(z, f(z)) \in \mathbb{C}^2 : z \in \overline{\mathbb{D}}\} \subset M$. Since $f(e^{i\theta}) = 0$ for $\theta \in \mathbb{R}$, by using maximum modulus theorem, we infer that $\widehat{K}$ contains the analytic disc $\{(z,0) \in \mathbb{C}^2 : z \in \overline{\mathbb{D}}\}$. Hence, K is not polynomially convex. Some sufficient conditions for polynomial convexity of totally-real discs in $\mathbb{C}^2$, i.e., the compact subset $\{(z, f(z)) \in \mathbb{C}^2; z \in \overline{\mathbb{D}}\}$ of a totally real graph in $\mathbb{C}^2$, in terms of the graphing function f, are available in the literature (see [4, 14, 15], and [18] for a nice survey). Forstnerič [7] showed that $\mathcal{C}^2$-perturbation of totally-real polynomially convex compact subset is polynomially convex. Løw and Wold [13] brought the smoothness down to $\mathcal{C}^1$. From these result we know that $\mathcal{C}^1$-perturbation of any compact subset lying in a totally-real subspace in $\mathbb{C}^n$ is polynomially convex, but there are no general results for compact subsets of $\mathbb{C}^n$, $n > 2$, that we are aware of. Therefore, it seems interesting to know the conditions under which a compact subset of a totally-real submanifold of $\mathbb{C}^n$ is polynomially convex. In this paper we report the results of our investigations on this question.

We now present the main results of this paper. Let K be a compact subset of a totally-real graph over $\mathbb{C}^n$ in $\mathbb{C}^{2n}$. In this case we present a necessary and sufficient condition for polynomial convexity of the given compact K in terms of the graphing functions:

Theorem 1.2. *Let $f^1, \dots, f^n : \mathbb{C}^n \longrightarrow \mathbb{C}$ be $\mathcal{C}^2$-smooth functions such that, writing $F = (f^1, \dots, f^n)$, the graph $\mathsf{Gr}_{\mathbb{C}^n}(F)$ is a totally real submanifold of $\mathbb{C}^{2n}$. Then, a compact subset K of $\mathsf{Gr}_{\mathbb{C}^n}(F)$ is polynomially convex if and only if there exists a $\Psi \in \mathsf{psh}(\mathbb{C}^{2n})$ such that*

$$K \subset \omega \subset G := \left\{(z,w) \in \mathbb{C}^n \times \mathbb{C}^n : \sum_{\nu=1}^{n} |f^\nu(z) - w_\nu| < \frac{m(z)}{2L(z)}\right\},$$

where

$$\omega := \{(z,w) \in \mathbb{C}^n \times \mathbb{C}^n : \Psi(z,w) < 0\},$$

$$L(z) := \max_{\nu \leq n} \left[\sup_{||v||=1} |\mathfrak{L}f^\nu(z;v)|\right]; \quad \textit{and}$$

$$m(z) := \inf_{||v||=1} \left(\sum_{\nu=1}^{n} \left|\frac{\partial f^\nu}{\partial \overline{z_1}}(z)\overline{v_1} + \dots + \frac{\partial f^\nu}{\partial \overline{z_n}}(z)\overline{v_n}\right|^2\right).$$

Here, and in what follows, $\mathfrak{L}f(z;.)$ denotes the Levi-form of a $\mathcal{C}^2$-smooth function f at z. We now make a couple of remarks that will aid the understanding of the statement of Theorem 1.2.

Remark 1.3. The radius of the tube-like set

$$G = \left\{ (z,w) \in \mathbb{C}^n \times \mathbb{C}^n : \sum_{\nu=1}^{n} |f^\nu(z) - w_\nu| < \frac{m(z)}{2L(z)} \right\}$$

may vary pointwise in $\mathbb{C}^{2n}$ but the totally-real assumption on the graph $\mathsf{Gr}_{\mathbb{C}^n}(F)$ ensures that $m(z) \neq 0$ for $z \in \mathbb{C}^n$ (see Lemma 2.6 in Section 2). Therefore, the tube-like set G is a nonempty open subset of $\mathbb{C}^{2n}$ containing the compact K.

Remark 1.4. We observe that if, in addition, we assume that the functions $f_1, \dots, f_n$ in Theorem 1.2 are pluriharmonic, then the above tubular neighbourhood has infinite radius at each point of $\mathsf{Gr}_{\mathbb{C}^n}(F)$. We just choose ω to be a suitable polydisc containing K, $K \subset \mathsf{Gr}_{\mathbb{C}^n}(F)$, such that the conditions of Theorem 1.2 are satisfied. Thus, any compact subset of such a graph is polynomially convex.

We would like to mention that Theorem 1.2 is a generalization of a result [6, Theorem 6.1]—which characterizes polynomial convexity of graphs over polynomially convex subset $\overline{\Omega}$, where Ω is a bounded domain in $\mathbb{C}^n$—in author's dissertation.

We now consider the case when the compact K lies in a totally-real submanifold which is a level set of a $\mathcal{C}^2$-smooth submersion on $\mathbb{C}^n$.

Theorem 1.5. *Let M be a $\mathcal{C}^2$-smooth totally-real submanifold of $\mathbb{C}^n$ of real dimension k such that $M := \rho^{-1}\{0\}$, where $\rho := (\rho_1, \dots, \rho_{2n-k})$ is a submersion from $\mathbb{C}^n$ to $\mathbb{R}^{2n-k}$, and K is a compact subset of M. Then K is polynomially convex if and only if there exists $\Psi \in \mathsf{psh}(\mathbb{C}^n)$ such that*

$$K \subset \omega \subset \Omega := \left\{ z \in \mathbb{C}^n : \sum_{l=1}^{2n-k} |\rho_l(z)| < \frac{m(z)}{L(z)} \right\},$$

where

$$\omega := \{z \in \mathbb{C}^n : \Psi(z) < 0\},$$
$$L(z) := \max_{l \le 2n-k} \left(\sup_{||v||=1} |\mathfrak{L}\rho_l(z,v)| \right); \quad \textit{and}$$
$$m(z) := \inf_{||v||=1} \sum_{l=1}^{2n-k} \left| \sum_{j=1}^{n} \frac{\partial \rho_l}{\partial \overline{z_j}}(z) v_j \right|^2 .$$

Remark 1.6. It is well known that a compact subset $K \subset \mathbb{C}^n$ is polynomially convex if and only if, for *every* neighbourhood U, there exists a polynomial polyhedron that contains the compact and lies inside U. From Theorem 1.5 we conclude that for a compact subset K of a totally-real submanifold of $\mathbb{C}^n$ to be polynomially convex it suffices that, for *a single* fixed neighbourhood U depending on the defining equations, we can find a polynomial polyhedron that contains K and is contained in U.

As in Remark 1.3, the fact that $\rho^{-1}\{0\}$ is totally real ensures that $m(z) \neq 0$, for all $z \in K$ (see Lemma 2.5 in Section 2); the set $\Omega = \left\{ z \in \mathbb{C}^n : \sum_{l=1}^{2n-k} |\rho_l(z)| < \dfrac{m(z)}{L(z)} \right\}$ is an open set containing K.

Remark 1.7. We should point out that we can use Theorem 1.5 for compact subsets of totally-real graphs $\mathsf{Gr}_{\mathbb{C}^n}(F)$ with the defining functions

$$\rho_{2j-1}(z,w) = \mathsf{Re}(w_{2j-1}) - \mathsf{Re}(f_{2j-1}(z)), j = 1, \dots, n,$$
$$\rho_{2j}(z,w) = \mathsf{Im}(w_{2j}) - \mathsf{Im}(f_{2j}(z)), j = 1, \dots, n.$$

It might be possible to deduce Theorem 1.2 from Theorem 1.5, but it is not easy to see. The tube like domain Ω that appears in Theorem 1.5 with the above defining functions seems to be different from the tube-like domain G in Theorem 1.2 and $G \not\subset \Omega$.

Before proceeding further let us discuss briefly the main ideas behind the proofs of both the theorems.

- Firstly, it will be shown that a compact patch of the totally-real submanifold containing K is holomorphically convex in the tube-like domain G and Ω in Theorems 1.2 and 1.5 respectively. For that we start with a function $u(z,w) := \sum_{\nu=1}^{n} |w_\nu - f^\nu(z)|^2$ for Theorem 1.2; and $u(z,w) := \sum_{l=1}^{2n-k} \rho_l^2(z)$ for Theorem 1.5. The specific geometric assumptions on the graphing functions or defining functions gives the terms of quantities $L(z)$ and $m(z)$ so that u is plurisubharmonic in G and Ω respectively.
- Secondly, since a sublevel set ω of a plurisubharmonic function on $\mathbb{C}^n$ is Runge and the compact patch is holomorphically convex in ω, the compact patch containing K is polynomially convex. Then an approximation result gives us the polynomial convexity of K.

We conclude the section with an observation about polynomial convexity of compact subsets that lie in an arbitrary totally-real submanifold of $\mathbb{C}^n$, and not *just* a zero set of a submersion defined on *all of* $\mathbb{C}^n$.

Remark 1.8. An abstract result analogous to Theorem 1.5 holds for compacts that lie in any arbitrary totally-real submanifold of $\mathbb{C}^n$. The construction of a suitable tubular neighbourhood that will replace the tube-like neighbourhood in Theorem 1.5 is the main obstacle, which can be overcome by using partitions of unity. In this case, locally, we have real valued $\mathcal{C}^2$-smooth functions $\rho_1, \dots, \rho_{2n-k}$ such that the submanifold can be viewed locally as the zero set of a submersion $\rho = (\rho_1, \dots, \rho_{2n-k})$; thus, locally we get a neighbourhood defined in terms of $\rho_1, \dots, \rho_{2n-k}$ as in Theorem 1.5. The problem with the result that we will end up with is that, since the tube ω would be given in terms of local data and (highly non-unique) cut-off functions, it would be merely an abstraction. Of course, highly abstract characterisations of polynomial convexity, in the language of uniform algebras, *already* exist—but hard to check. The point of this paper is to begin with some natural overarching assumption and derive characterisations for polynomial convexity that are checkable. A couple of examples of totally-real submanifolds are given in Section 5 as applications of Theorem 1.2 and Theorem 1.5. We keep the discussion of generalizing Theorem 1.5 to more general Stein manifolds for a future project.

2. Technical Results

In this section, we prove some results that will be used in the proofs of our theorems: Lemma 2.2, a result about closed subsets of polynomially convex compact sets (Lemma 2.3), and two results characterizing when a submanifold of $\mathbb{C}^n$ is totally real (Lemma 2.6 and Lemma 2.5). We begin by stating a basic but nontrivial result by Hörmander [11, Theorem 4.3.4] that will be used several times in this paper.

Result 2.1 (Hörmander, Lemma 4.3.4, [11])**.** *Let K be a compact subset of a pseudoconvex open set $\Omega \subset \mathbb{C}^n$. Then $\widehat{K}_\Omega = \widehat{K}_\Omega^P$, where $\widehat{K}_\Omega^P := \{z \in \Omega : u(z) \le \sup_K u \ \forall u \in \mathsf{psh}(\Omega)\}$ and $\widehat{K}_\Omega := \{z \in \Omega : |f(z)| \le \sup_{z \in K} |f(z)| \ \forall f \in \mathcal{O}(\Omega)\}$.*

We note that if $\Omega = \mathbb{C}^n$ then Result 2.1 says that the polynomially convex hull of K is equal to the plurisubharmonically convex hull of K.

Next we prove a couple of lemmas, which have vital roles in the proofs of our theorems, about the polynomially convex hull of general compact subsets $\mathbb{C}^n$.

Lemma 2.2. *Let K be a compact set in $\mathbb{C}^n$, and let ϕ be a plurisubharmonic function on $\mathbb{C}^n$ such that $K \subset \Omega$, where $\Omega := \{z \in \mathbb{C}^n : \phi(z) < 0\}$. Suppose there exists a non-negative function $v \in \mathsf{psh}(\Omega)$ such that $v(z) = 0 \ \forall z \in K$. Then $\widehat{K} \subseteq v^{-1}\{0\}$.*

Proof. We are given that $K \Subset \Omega = \{z \in \mathbb{C}^n : \phi(z) < 0\}$. Hence, by the Result 2.1, $\widehat{K}_\Omega^P = \widehat{K}_\Omega \subset \Omega$. Upper-semicontinuity of ϕ gives $\widehat{K} \Subset \Omega$. Since Ω is a pseudoconvex domain, $\widehat{K}_\Omega^P \Subset \Omega$. Therefore, $\widehat{K}_\Omega^P \subset v^{-1}\{0\}$, which implies $\widehat{K}_\Omega \subset v^{-1}\{0\}$. Since Ω is also a Runge domain, $\widehat{K} = \widehat{K}_\Omega$. Hence, $\widehat{K} \subset v^{-1}\{0\}$. ☐

Lemma 2.3. *Let K be a compact polynomially convex subset of a totally-real submanifold of $\mathbb{C}^n$. Then any closed subset of K is polynomially convex.*

Proof. Since K is a polynomially convex subset of a totally-real submanifold of $\mathbb{C}^n$, we apply Result 1.1 to get $\mathcal{P}(K) = \mathcal{C}(K)$. Let L be a closed subset of K. By Tietze extension theorem, $\mathcal{C}(K)|_L = \mathcal{C}(L)$. Since $\mathcal{P}(K)|_L \subset \mathcal{P}(L) \subset \mathcal{C}(L)$, we have $\mathcal{P}(L) = \mathcal{C}(L)$. Hence, L is polynomially convex. ☐

We now state a result due to Oka (see [11, Lemma 2.7.4]) that gives us one direction of the implications in both the theorems in this paper.

Result 2.4 (Oka)**.** *A compact subset K of $\mathbb{C}^n$ is polynomially convex if and only if for every neighbourhood U of K there exist a polynomial polyhedron P such that*

$$K \subset P \subset U.$$

Recall that a polynomial polyhedron is the set $\{z \in \mathbb{C}^n : |p_j(z)| \le 1, \ j = 1, \dots, m\}$ for finitely many polynomials $p_1, \dots, p_m$.

Let M be a $\mathcal{C}^1$-smooth real submanifold of $\mathbb{C}^n$ of real dimension k, $k \le n$. For each $p \in M$ there exists a neighbourhood U_p of p in $\mathbb{C}^n$ and $\mathcal{C}^2$-smooth real-valued functions $\rho_1, \rho_2, \dots, \rho_{2n-k}$ such that

$$U_p \cap M = \{z \in U_p : \rho_1(z) = \rho_2(z) = \dots = \rho_{2n-k}(z) = 0\},$$

where $\rho = (\rho_1, \dots, \rho_{2n-k})$ is a submersion. With these notations we now state the following lemma.

Lemma 2.5. *M is totally real at $p \in M$ if and only if the matrix A_p is of rank n, where*

$$A_p := \begin{pmatrix} \dfrac{\partial \rho_1}{\partial \overline{z_1}}(p) & \cdots \dfrac{\partial \rho_1}{\partial \overline{z_n}}(p) \\ \dfrac{\partial \rho_2}{\partial \overline{z_1}}(p) & \cdots \dfrac{\partial \rho_2}{\partial \overline{z_n}}(p) \\ \vdots & \vdots \\ \dfrac{\partial \rho_{2n-k}}{\partial \overline{z_1}}(p) & \cdots \dfrac{\partial \rho_{2n-k}}{\partial \overline{z_n}}(p) \end{pmatrix}.$$

Proof. Viewing $\mathbb{C}^n$ as $\mathbb{R}^{2n}$, the tangent space T_pM can be described as:

$$T_pM = \left\{v \in \mathbb{R}^{2n} : D\rho(p)v = 0\right\}.$$

We first assume that M is totally real at $p \in M$. We will show that the rank of A_p is n. Suppose the matrix A_p has rank less than n. Without loss of generality, we may assume that the rank of A_p is $n-1$. Hence, there exists $v = (v_1, \dots, v_n) \in \mathbb{C}^n \setminus \{0\}$ such that

$$A_p v = 0.$$

This implies that the system of linear equation

$$\sum_{j=1}^{n} \frac{\partial \rho_l}{\partial \bar{z}_j}(p) v_j = 0, \; l = 1, \dots, 2n-k, \tag{2.1}$$

has a nonzero solution. Viewing $v_j = v'_j + iv''_j$, $j = 1, \dots, n$, and writing the system of equations (2.1) in terms of real coordinates, for each $l = 1, \dots, 2n-k$, we obtain that the complex equation

$$\sum_{j=1}^{n} \left(\frac{\partial \rho_l}{\partial x_j}(p) + i \frac{\partial \rho_l}{\partial y_j}(p) \right) (v'_j + iv''_j) = 0$$

is equivalent to following systems of real equations

$$\sum_{j=1}^{n} \left(\frac{\partial \rho_l}{\partial x_j}(p) v'_j - \frac{\partial \rho_l}{\partial y_j}(p) v''_j \right) = 0; \tag{2.2}$$

$$\sum_{j=1}^{n} \left(\frac{\partial \rho_l}{\partial x_j}(p) v''_j + \frac{\partial \rho_l}{\partial y_j}(p) v'_j \right) = 0. \tag{2.3}$$

In view of (2.3), we get that the vector $v = (v'_1, v''_2, \dots, v'_n, v''_n)$ lies in T_pM, and the equations in (2.2) ensure that $iv \in T_pM$ (viewing $v = (v'_1 + iv''_1, \dots, v'_n + iv''_n) \in \mathbb{C}^n$). This is a contradiction to the fact that M is totally real at p.

For the converse, assume the matrix A_p has rank n. We show that M is totally real at $p \in M$. Suppose M is not totally real at p, i.e., there exists a $v \in T_pM$, $v \neq 0$, such that $iv \in T_pM$. This implies that equations (2.3) and (2.2) hold. Hence, $A_p v = 0$, which contradicts the assumption that rank of A_p is n. Hence, M is totally real at p. □

Next we state a lemma that gives a characterization for a graph in $\mathbb{C}^{2n}$, using the graphing functions, to have complex tangents.

Lemma 2.6. *Let $f^1, \dots, f^n : \mathbb{C}^n \longrightarrow \mathbb{C}$ be $\mathcal{C}^1$-smooth functions. Let $M := \{(z, w) \in \mathbb{C}^{2n} : w_\nu = f^\nu(z),\ \nu = 1, \dots, n\}$. Let $P := (a, f^1(a), \dots, f^n(a)) \in M$. Then, M has a complex tangent at $P \in M$ if and only if there exists a vector $(v_1, \dots, v_n) \in \mathbb{C}^n \setminus \{0\}$ such that*

$$\sum_{j=1}^{n} v_j \frac{\partial f^\nu}{\partial \bar{z}_j}(a) = 0 \quad \forall \nu = 1, \dots, n.$$

Proof. The proof follows from the following fact due to Wermer [23].

Fact. *Let $h_1, \dots, h_m$ be $\mathcal{C}^1$-smooth complex valued functions defined in a neighbourhood U of $0 \in \mathbb{R}^k$ such that the function $h := (h_1, \dots, h_m)$ is a regular map on U into $\mathbb{C}^m$. Let $S := h(U)$. Then, S is totally real at $h(0)$ if and only if the complex rank of the matrix $\left(\dfrac{\partial h_j}{\partial x_i}(0)\right)_{i,j}$ is k.*

□

3. The proof of Theorem 1.2

We begin the proof by constructing a tube-like neighbourhood of the graph and a non-negative plurisubharmonic function defined in it, which vanishes on the graph. This constitutes Step I. Further steps then lead us to showing the desired compact to be polynomially convex on the basis of our construction in the first step.

Step I: *Constructing a tube-like neighbourhood G of the graph and a plurisubharmonic function u on G.*

In this case we consider the defining functions:

$$u_\nu(z,w) = |w_\nu - f^\nu(z)|, \quad \nu = 1, \ldots, n.$$

and

$$u(z,w) := \sum_{\nu=1}^{n} u_\nu^2(z,w).$$

We obtain the Levi form:

$$\begin{aligned}\mathfrak{L}u(\bullet;V) = &\sum_{\nu=1}^{n}\left((f^\nu - w_\nu)\sum_{j,k=1}^{n}\overline{\frac{\partial^2 f^\nu}{\partial\overline{z_j}\partial z_k}\overline{v_j}v_k} + (\overline{f^\nu} - \overline{w_\nu})\sum_{j,k=1}^{n}\frac{\partial^2 f^\nu}{\partial z_j\partial\overline{z_k}}v_j\overline{v_k}\right)\\ &+\sum_{j,k=1}^{n}\left(\sum_{\nu=1}^{n}\frac{\partial f^\nu}{\partial z_j}\overline{\frac{\partial f^\nu}{\partial z_k}}\right)v_j\overline{v_k} + \sum_{j,k=1}^{n}\left(\sum_{\nu=1}^{n}\frac{\partial f^\nu}{\partial\overline{z_k}}\overline{\frac{\partial f^\nu}{\partial\overline{z_j}}}\right)v_j\overline{v_k}\\ &-\sum_{j,k=1}^{n}\frac{\partial f^k}{\partial z_j}v_j\overline{t_\nu} - \sum_{j,k=1}^{n}\overline{\frac{\partial f^j}{\partial z_k}v_k\overline{t_j}} + \sum_{\nu=1}^{n}|t_\nu|^2,\end{aligned}$$

where we denote $V = (v,t) = (v_1,\ldots,v_n,t_1,\ldots,t_n) \in \mathbb{C}^{2n}$. Swapping the subscripts j and k in the first sum in the second line above allows us to see that:

$$\begin{aligned}\mathfrak{L}u(\bullet;V) &= 2\sum_{\nu=1}^{n}\mathsf{Re}\big((\overline{f^\nu} - \overline{w_\nu})\mathfrak{L}f^\nu(z;v)\big) + \sum_{j,k=1}^{n}\left(\sum_{\nu=1}^{n}\frac{\partial f^\nu}{\partial z_j}\overline{\frac{\partial f^\nu}{\partial z_k}}\right)v_j\overline{v_k}\\ &\quad+\sum_{j,k=1}^{n}\left(\sum_{\nu=1}^{n}\frac{\partial f^\nu}{\partial\overline{z_k}}\overline{\frac{\partial f^\nu}{\partial\overline{z_j}}}\right)v_j\overline{v_k} - \sum_{j,k=1}^{n}\frac{\partial f^k}{\partial z_j}v_j\overline{t_k} - \sum_{j,k=1}^{n}\overline{\frac{\partial f^j}{\partial z_k}v_k\overline{t_j}} + \sum_{\nu=1}^{n}|t_\nu|^2.\\ &= 2\sum_{\nu=1}^{n}\mathsf{Re}\big((\overline{f^\nu} - \overline{w_\nu})\mathfrak{L}f^\nu(z;v)\big) + \sum_{\nu=1}^{n}\left|\frac{\partial f^\nu}{\partial z_1}v_1 + \cdots + \frac{\partial f^\nu}{\partial z_n}v_n - t_\nu\right|^2\\ &\qquad\qquad+\sum_{j,k=1}^{n}\left(\sum_{\nu=1}^{n}\frac{\partial f^\nu}{\partial\overline{z_k}}\overline{\frac{\partial f^\nu}{\partial\overline{z_j}}}\right)v_j\overline{v_k}\\ &= 2\sum_{\nu=1}^{n}\mathsf{Re}\big((\overline{f^\nu} - \overline{w_\nu})\mathfrak{L}f^\nu(z;v)\big) + \sum_{\nu=1}^{n}\left|\frac{\partial f^\nu}{\partial z_1}v_1 + \cdots + \frac{\partial f^\nu}{\partial z_n}v_n - t_\nu\right|^2\\ &\qquad\qquad+\sum_{\nu=1}^{n}\left|\frac{\partial f^\nu}{\partial\overline{z_1}}\overline{v_1} + \cdots + \frac{\partial f^\nu}{\partial\overline{z_n}}\overline{v_n}\right|^2\\ &\geq \sum_{\nu=1}^{n}\left|\frac{\partial f^\nu}{\partial\overline{z_1}}\overline{v_1} + \cdots + \frac{\partial f^\nu}{\partial\overline{z_n}}\overline{v_n}\right|^2 - 2\sum_{\nu=1}^{n}|f^\nu - w_\nu||\mathfrak{L}f^\nu(z;v)|.\end{aligned} \tag{3.1}$$

Let

$$L(z) := \max_{\nu} \left(\sup_{||v||=1} |\mathfrak{L} f^{\nu}(z; v)| \right),$$

and

$$m(z) := \inf_{||v||=1} \left(\sum_{\nu=1}^{n} \left| \frac{\partial f^{\nu}}{\partial \overline{z_1}} \overline{v_1} + \cdots + \frac{\partial f^{\nu}}{\partial \overline{z_n}} \overline{v_n} \right|^2 \right).$$

Since the graph $\mathsf{Gr}_{\mathbb{C}^n}(F)$ is totally real, by Lemma 2.6, we have $m(z) > 0$ for all $z \in \mathbb{C}^n$. Define

$$G := \left\{ (z, w) \in \mathbb{C}^{2n} : \sum_{\nu=1}^{n} |f^{\nu}(z) - w_{\nu}| < \frac{m(z)}{2L(z)} \right\}.$$

From (3.1), it is clear that u is strictly plurisubharmonic on G and $u^{-1}\{0\} = \mathsf{Gr}(f^1, ..., f^n)$. Since $\omega \subset G$ (by hypothesis), we have $u \in \mathsf{psh}(\omega)$ and $K \subset \mathsf{Gr}_{\mathbb{C}^n}(F) \subset u^{-1}\{0\}$.

Step II: *Showing that* $\widehat{K} \subset u^{-1}\{0\}$.

Since, by Step I, $u \in \mathsf{psh}(\omega)$ and $K \subset u^{-1}\{0\}$, all the conditions of Lemma 2.2 are fulfilled with given compact K, $v := u$ and $\phi := \Psi$. Hence, in view of Lemma 2.2, we obtain

$$\widehat{K} \subset u^{-1}\{0\}.$$

Step III: *Completing the proof.*

The aim of this step is to show that K is polynomially convex. After a suitable regularization of Ψ we may assume that Ψ is continous. We now consider $K_1 := \{(z, w) \in \mathsf{Gr}_{\mathbb{C}^n}(F) : \Psi(z, w) + \varepsilon \leq 0\}$, where

$$-\varepsilon := \sup_{K} \Psi(z, w).$$

Clearly, $K \subset K_1$. Thanks to the fact that $K \subset \omega = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^n : \Psi(z, w) < 0\}$, we get that $\varepsilon > 0$. Ψ is plurisubharmonic in $\mathbb{C}^{2n}$,

$$\widehat{K_1} \subset \{(z, w) \in \mathbb{C}^{2n} : \Psi(z, w) < 0\} = \omega \subset G.$$

By Lemma 2.2, with the compact K_1, $\Omega := G$ $v := u$ and $\phi := \Psi$, we conclude that $\widehat{K_1} \subset u^{-1}\{0\} = M$. Hence, K_1 is polynomially convex. Using Lemma 2.3, we conclude that K is polynomially convex.

The converse follows from Result 2.4.

4. Proof of Theorem 1.5

Our proof of Theorem 1.5 follows in lines similar to that of Theorem 1.2. Again, using the defining equations, we will construct a nonnegative plurisubharmonic function in a tubular neighbourhood of the given compact subset K.

Proof of Theorem 1.5. As before, we divide the proof in three steps.

Step I. *Existence of a plurisubharmonic function* u *on a neighbourhood of* K *with* $K \subset u^{-1}\{0\}$.

Let us define the following function:

$$u(z) := \sum_{l=1}^{2n-k} \rho_l^2(z).$$

We now compute the Levi-form for the above function u. For that, We have

$$\frac{\partial^2 u}{\partial z_j \partial \overline{z_k}}(z) = 2\sum_{l=1}^{2n-k} \rho_l(z) \frac{\partial^2 \rho_l}{\partial z_j \partial \overline{z_k}}(z) + 2\sum_{l=1}^{2n-k} \frac{\partial \rho_l}{\partial z_j}(z) \frac{\partial \rho_l}{\partial \overline{z_k}}(z). \tag{4.1}$$

Hence, the Levi-form of u:

$$\begin{aligned}
\mathcal{L}u(z,v) &= \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z_k}}(z) v_j \overline{v_k} \\
&= 2\sum_{l=1}^{2n-k} \sum_{j,k=1}^{n} \rho_l(z) \frac{\partial^2 \rho_l}{\partial z_j \partial \overline{z_k}}(z) v_j \overline{v_k} + 2\sum_{l=1}^{2n-k} \left| \sum_{j=1}^{n} \frac{\partial \rho_l}{\partial \overline{z_j}}(z) v_j \right|^2 \\
&= 2\sum_{l=1}^{2n-k} \rho_l(z) \mathcal{L}\rho_l(z,v) + 2\sum_{l=1}^{2n-k} \left| \sum_{j=1}^{n} \frac{\partial \rho_l}{\partial \overline{z_j}}(z) v_j \right|^2 \\
&\geq 2\sum_{l=1}^{2n-k} \left| \sum_{j=1}^{n} \frac{\partial \rho_l}{\partial \overline{z_j}}(z) v_j \right|^2 - 2\sum_{l=1}^{2n-k} |\rho_l(z)| |\mathcal{L}\rho_l(z,v)|. \qquad (4.2)
\end{aligned}$$

Let us define the following set

$$\Omega := \left\{ z \in \mathbb{C}^n : \sum_{l=1}^{2n-k} |\rho_l(z)| < \frac{m(z)}{L(z)} \right\},$$

where

$$L(z) := \max_{l} \left(\sup_{||v||=1} |\mathcal{L}\rho_l(z,v)| \right),$$

and

$$m(z) := \inf_{||v||=1} \sum_{l=1}^{2n-k} \left| \sum_{j=1}^{n} \frac{\partial \rho_l}{\partial \overline{z_j}}(z) v_j \right|^2.$$

Since M is totally real, by Lemma 2.5, we get that $m(z) > 0$ for $z \in M$. From (4.2), we obtain that

$$\mathcal{L}u(z,v) \geq 0, \text{ for all } z \in \Omega,$$

Hence, u is plurisubharmonic in Ω and $K \subset u^{-1}\{0\}$.

Step II. *Showing that* $\widehat{K} \subset u^{-1}\{0\}$.

Let us denote $\omega := \{z \in \mathbb{C}^n : \phi(z) < 0\}$. By the assumption u is plurisubharmonic in ω. Invoking Lemma 2.2 again with $\Omega := \omega$, $v := u$, we get that

$$\widehat{K} \subset u^{-1}\{0\}.$$

Step III. *Completing the proof.*

As in the proof of Theorem 1.2 we consider

$$K_1 := M \cap \{z \in \mathbb{C}^n : \phi(z) + \varepsilon \leq 0\},$$

where $-\varepsilon = \sup_K \phi(z)$. The remaining part of the proof goes in the same way as in Step III of the proof of Theorem 1.2.

As before, the converse follows from Result 2.4 □

5. Examples

In this section we provide a couple of examples of totally-real submanifolds of $\mathbb{C}^2$: the first one is given in Hörmander-Wermer [10].

Example 5.1. We consider the graph $K = \{(z, f(z)) \in \mathbb{C}^2 : z \in \overline{\mathbb{D}}\}$ over the closed unit disc $\overline{\mathbb{D}}$, where $f(z) = -(1+i)\overline{z} + iz\overline{z}^2 + z^2\overline{z}^3$. It is shown in [10], by attaching an analytic disc to $\{(z, f(z)) : |z| = 1\} \subset K$, that K is not polynomially convex. Here we focus on the closed subsets of the compact K of the form:

$$K_r := \{(z, f(z)) \in \mathbb{C}^2 : |z| \leq r\}.$$

Since K is a subset of a totally-real submanifold $M = \{(z, f(z)) : z \in \mathbb{C}\}$, we already know that there exists an $r > 0$ such that $\widehat{K_r} = K_r$. Here a range for r is deduced for which K_r is polynomially convex. We use Theorem 1.2 to show that K_r is polynomially convex for all $r \in [0, 1/2]$.

Let us first compute:

$$\frac{\partial f}{\partial \overline{z}}(z) = -(1+i) + 2i|z|^2 + 3|z|^4,$$
$$\frac{\partial^2 f}{\partial z \partial \overline{z}}(z) = 2\overline{z}(i + 3|z|^2).$$

Hence, in the notation of Theorem 1.2, we have

$$L(z) = \left|\frac{\partial^2 f}{\partial z \partial \overline{z}}(z)\right| = 2|z|\sqrt{1 + 9|z|^4}$$
$$m(z) = \left|\frac{\partial f}{\partial \overline{z}}(z)\right|^2 = 9|z|^8 - 2|z|^4 - 4|z|^2 + 2. \tag{5.1}$$

We get a neighbourhood of K as

$$\Omega := \left\{(z, w) \in \mathbb{C}^2 : |w - f(z)| < \frac{9|z|^8 - 2|z|^4 - 4|z|^2 + 2}{2|z|\sqrt{1 + 9|z|^4}}\right\},$$

in which $u(z, w) := |w - f(z)|^2$ is plurisubharmonic. We now note that the function $h(r) := 9r^4 - 2r^2 - 4r + 2$ is monotonically decreasing in the interval $[0, 1/3]$. Hence, from (5.1) we have

$$\inf_{|z| \leq 1/2} m(z) = h(1/4) = 233/256.$$

Consider another function

$$g(s) := 2s\sqrt{1 + 9s^4}.$$

A simple computation shows us that g is strictly increasing in [0,1/2]. Hence, we have

$$\sup_{|z| \leq 1/2} L(z) = g(1/2) = 5/4.$$

Hence, we get that the open set $\widetilde{\Omega} := \{(z, w) \in \mathbb{C}^2 : |z| < 1/2 + \delta, |w - f(z)| < 233/320 + \varepsilon\} \subset \Omega$ for sufficiently small $\delta > 0$ and $\varepsilon > 0$. We now consider the following function

$$\phi(z, w) := |w + (1+i)\overline{z}|^2.$$

Since the quadratic for corresponding to the complex hessian of ϕ is:

$$\frac{\partial^2 \phi}{\partial z \partial \overline{z}}|u|^2 + \frac{\partial^2 \phi}{\partial z \partial \overline{w}}u\overline{v} + \frac{\partial^2 \phi}{\partial w \partial \overline{z}}v\overline{u} + \frac{\partial^2 \phi}{\partial w \partial \overline{w}}|v|^2 = |u|^2 + 2|v|^2 > 0 \ \forall (u, v) \in \mathbb{C}^2 \setminus \{0\}.$$

Note that ϕ is a strongly plurisubharmonic function on $\mathbb{C}^2$. We have

$$\sup_{|z|\leq 1/2} |z|^3\sqrt{1+|z|^4} = \frac{\sqrt{17}}{32}.$$

Consider the function $\psi : \mathbb{C}^2 \to \mathbb{R}$ defined by

$$\psi(z,w) := \phi(z,w) - k^2,$$

where $k = \dfrac{233-10\sqrt{17}}{320} - \widetilde{\varepsilon}$ for sufficiently small $\widetilde{\varepsilon}$. Clearly, ψ is also strongly plurisubharmonic in $\mathbb{C}^2$.

Claim:

$$K_r \subset \{\psi(z,w)<0\} \cap \{(z,w)\in\mathbb{C}^2 : |z| < 1/2+\delta\} \subset \widetilde{\Omega},$$

for some sufficiently small $\delta > 0$.

Let us first proof the second part. Suppose $(z,w)\in\mathbb{C}^2$ such that $|z|\leq 1/2$ such that $\psi(z,w)<0$. This implies:

$$\begin{aligned}
&\phi(z,w) < k^2\\
\implies&|w+(1+i)\overline{z}| < k\\
\implies&|w+(1+i)\overline{z}| + \frac{10\sqrt{17}}{320} < \frac{233}{320}\\
\implies&|w+(1+i)\overline{z}| + \sup_{|z|\leq 1/2}|(i+|z|^2)\overline{z}||z|^2 < \frac{233}{320}\\
\implies&|w-f(z)| \leq |w+(1+i)\overline{z}| + \sup_{|z|\leq 1/2}|(i+|z|^2)\overline{z}||z|^2 < \frac{233}{320}.
\end{aligned}$$

Hence, for sufficiently small $\delta>0$, we have

$$\{\psi(z,w)<0\} \cap \{(z,w)\in\mathbb{C}^2 : |z|<1/2+\delta\} \subset \widetilde{\Omega}.$$

We now show the first part, i.e., we need to show that for $|z|\leq 1/2$,

$$\psi(z,f(z)) < 0.$$

We have

$$\begin{aligned}
\phi(z,f(z) &= |f(z)+(1+i)\overline{z}|^2\\
&= |z|^6(1+|z|^4).
\end{aligned}$$

Since $h(r) := r^3(1+r^2)$ is an increasing function on $[0,1/2]$, we have

$$\sup_{|z|\leq 1/2} |z|^6(1+|z|^4) = \frac{17}{1024}.$$

Hence, we obtain

$$\sup_{|z|\leq 1/2} \phi(z,f(z)) < k^2.$$

Hence, $K_r \subset \{\psi(z,w)<0\} \cap \{(z,w)\in\mathbb{C}^2 : |z|<1/2+\delta\}$. Therefore, by using Theorem 1.2, we conclude that $\widehat{K_r} = K_r$ for all $r\in[0,1/2]$.

Remark 5.2. An another example of similar type to Example 5.1 is given in [8]. The totally-real graph in [8] is:

$$M := \{(z, w) \in \mathbb{C}^2 : w = g(z)\},$$

where $g(z) = \overline{z}(|z|^2 - 1)e^{i|z|^2}$. Let $K := M \cap \{(z, w) \in \mathbb{C}^2 : z \in \overline{\mathbb{D}}\}$. An analytic disc $\phi : \overline{\mathbb{D}} \to \mathbb{C}^2$ defined by $\phi(z) = (z, 0)$ attached to K. After a computation analogous to that in Example 5.1 and applying Theorem 1.2, we can find that for every $r \in (0, 1/16\sqrt{11})$, K_r is polynomially convex.

Example 5.3. Let us consider the following graph in $\mathbb{C}^2$ over $\mathbb{R}^2$:

$$M = \{(x_1 + ic(x_1^2 + x_2^3), x_2 + id(x_2^2 + x_1^3)) \in \mathbb{C}^2 : x_1, x_2 \in \mathbb{R}\},$$

where $0 \leq c, d \leq \dfrac{1}{60}$. We show that the compact $K := M \cap \overline{\mathbb{D}^2}$ is polynomially convex. In this case we have $\rho := (\rho_1, \rho_2)$, where

$$\rho_1(z_1, z_2) := \frac{1}{2i}(z_1 - \overline{z_1}) - \frac{c}{4}\left((z_1 + \overline{z_1})^2 + \frac{1}{2}(z_2 + \overline{z_2})^3\right)$$
$$\rho_2(z_1, z_2) := \frac{1}{2i}(z_2 - \overline{z_2}) - \frac{d}{4}\left(\frac{1}{2}(z_2 + \overline{z_2})^2 + (z_1 + \overline{z_1})^3\right).$$

Clearly $K = \rho^{-1}\{0\} \cap \overline{\mathbb{D}^2}$. Using the notation $z_j = x_j + iy_j$, $j = 1, 2$, we compute:

$$\frac{\partial \rho_1}{\partial \overline{z_1}}(z) = i/2 - cx_1, \qquad \frac{\partial \rho_2}{\partial \overline{z_2}}(z) = i/2 - dx_2,$$
$$\frac{\partial \rho_1}{\partial \overline{z_2}}(z) = -\frac{3cx_2^2}{2}, \qquad \frac{\partial \rho_2}{\partial \overline{z_1}}(z) = -\frac{3dx_1^2}{2},$$
$$\frac{\partial^2 \rho_1}{\partial z_1 \partial \overline{z_1}}(z) = -c/2, \qquad \frac{\partial^2 \rho_1}{\partial z_1 \partial \overline{z_2}}(z) = 0, \qquad \frac{\partial^2 \rho_1}{\partial z_2 \partial \overline{z_2}}(z) = -\frac{3cx_2}{2},$$
$$\frac{\partial^2 \rho_2}{\partial z_1 \partial \overline{z_1}}(z) = -\frac{3dx_1}{2}, \qquad \frac{\partial^2 \rho_2}{\partial z_1 \partial \overline{z_2}}(z) = 0, \qquad \frac{\partial^2 \rho_2}{\partial z_2 \partial \overline{z_2}}(z) = -d/2.$$

From the above computation we get that:

$$\begin{aligned} L(z) &= \max_{l=1,2}\left(\sup_{||v||=1}\left|\frac{\partial^2 \rho_l}{\partial z_1 \partial \overline{z_1}}(z)|v_1|^2 + 2\mathsf{Re}\left(\frac{\partial^2 \rho_l}{\partial z_1 \partial \overline{z_2}}(z)v_1\overline{v_2}\right) + \frac{\partial^2 \rho_l}{\partial z_2 \partial \overline{z_2}}(z)|v_2|^2\right|\right) \\ &\leq 2\max\{c, d\} \\ m(z) &= \inf_{||v||=1}\left|\left(\frac{i}{2} - cx_1\right)v_1 - \frac{3cx_2^2}{2}v_2\right|^2 + \left|\left(\frac{i}{2} - dx_2\right)v_2 - \frac{3dx_2^2}{2}v_1\right|^2. \end{aligned}$$

Hence, the neighbourhood

$$\Omega = \left\{z \in \mathbb{C}^2 : |\rho_1(z)| + |\rho_2(z)| < \frac{m(z)}{L(z)}\right\}$$

of K contains the open set

$$\left\{z \in \mathbb{C}^n : |y_1 - c(x_1^2 + x_2^3)| + |y_2 - d(x_2^2 + x_1^3)| < \frac{5}{72\max\{c, d\}}\right\},$$

and $u(z) := \rho_1^2(z) + \rho_2^2(z)$ is plurisubharmonic in Ω. Since the constants $c, d < 1/60$, we get that

$$K \Subset D(0; 1 + \varepsilon) \times D(0; 1 + \varepsilon) \subset \Omega,$$

where $0 < \varepsilon < 1/100$. Therefore, by applying Theorem 1.5, we conclude that K is polynomially convex.

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References

[1] J. T. Anderson, A. J. Izzo and J. Wermer, *Polynomial approximation on three-dimensional real-analytic submanifolds of* $\mathbb{C}^n$, Proc. Amer. Math. Soc. **129** (2001), no. 8, 2395-2402.

[2] J. T. Anderson, A. J. Izzo and J. Wermer, *Polynomial approximation on real-analytic varieties in* $\mathbb{C}^n$, Proc. Amer. Math. Soc. **132** (2004), no. 5, 1495-1500.

[3] D. Catlin, *Boundary behavior of holomorphic functions on weakly pseudoconvex domains*, Dissertation, Princeton University, 1978.

[4] J. Duval, *Un exemple de disque polynômialement convexe*, Math. Ann. **281** (1988), no. 4, 583-588.

[5] T. W. Gamelin, *Uniform algebras*, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1969.

[6] S. Gorai, *Exploring polynomial convexity of certain classes of sets*, Ph. D. dissertation.

[7] F. Forstnerič, *Stability of polynomial convexity of totally real sets*, Proc. Amer. Math. Soc. **96** (1986), no. 3, 489-494.

[8] F. Forstnerič, *Intersections of analytic and smooth discs*(English Summary) *The Madison Symposium on Complex Analysis (Madison, WI, 1991)*, 235-244, Contemp. Math., **137**, Amer. Math. Soc., Providence, RI, 1992.

[9] F. R. Harvey and R. O. Wells Jr., *Holomorphic approximation and hyperfunction theory on a* $\mathcal{C}^1$ *totally real submanifold of a complex manifold*, Math. Ann. **197** (1972), 287-318.

[10] L. Hörmander and J. Wermer, *Uniform approximation on compact sets in* $\mathbb{C}^n$, Math. Scand. **23** (1968), 5-21.

[11] L. Hörmander, An Introduction to Complex Analysis in Several Variables, 3rd ed., North-Holland, Amsterdam, 1988.

[12] M. Lavrentiev, *Sur les fonctions d'une variable complexe représentables par des séries de polynomes*, Actualités Sci. Indust., no. 441 Hermann, Paris, 1936.

[13] E. Løw and E. F. Wold, *Polynomial convexity and totally real manifolds*, Complex Var. Elliptic Equ. **54** (2009), no. 3-4, 265-281.

[14] A. G. O'Farrell and K. J. Preskenis, *Uniform approximation by polynomials in two functions*, Math. Ann. **284** (1989), no. 4, 529-535.

[15] A. G. O'Farrell and K. J. Preskenis, *Approximation by polynomials in two diffeomorphisms*, Bull. Amer. Math. Soc. (N.S.) **10** (1984), no. 1, 105-107.

[16] A. G. O'Farrell, K. J. Preskenis and D. Walsh, *Holomorphic approximation in Lipschitz norms*, Contemp. Math. **32** (1984), 187-194.

[17] E. Poletsky, *Jensen measure and analytic multifunctions*, Ark. Mat. **42** (2004), no. 2, 335-352.

[18] A. Sanabria-García, *Polynomial hulls of smooth discs: a survey*, Irish Math. Soc. Bull. **45** (2000), 135-153.

[19] N. Sibony, *Some aspects of weakly pseudoconvex domains*, Several complex variables and complex geometry, Part 1 (Santa Cruz, CA, 1989), Proc. Sympos. Pure Math,, Vol. 52, Amer. Math. Soc., Providence, RI, 1991, pp. 199-231.

[20] E. L. Stout, *Holomorphic approximation on compact, holomorphically convex, real-analytic varieties*, Proc. Amer. Math. Soc. **134** (2006), no. 8, 2302-2308.

[21] J. Wermer, *Approximation on a disk*, Math. Ann. **155** (1964), 331-333.

[22] J. Wermer, *Polynomially convex disks*, Math. Ann. **158** (1965), 6-10.

[23] J. Wermer, *Polynomial approximation*, Proceeding of the Fifteenth Scandinavian Congress (Oslo, 1968), Lecture notes in Mathematics, Vol. 118 (Berlin), Springer, 1970, pp. 133-162.

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