INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH KOLKATA

Department of Mathematics and Statistics

End-semestral Examination, Autumn 2018

LINEAR ALGEBRA I (MA 2102)

Date: November 28, 2018 — Maximum Marks: 50 — Time: 1400 – 1700

Note: You can use well-known theorems taught in the class, but you need to write precise statement of the theorem that you are using.

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(1) (a) Prove that $\mathbb{R}^2$ and $\mathbb{C}$ are linearly isomorphic over $\mathbb{R}$ (that is, $\mathbb{R}$-linear isomorphism) via the map $\varphi : (x, y) \mapsto x + iy$.

(b) Let $L_A : \mathbb{R}^2 \to \mathbb{R}^2$ be the $\mathbb{R}$-linear map defined by $L_A(x) = Ax$, $x \in \mathbb{R}^2$, where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},\ a, b, c, d \in \mathbb{R}.$$

Show that the map $\widetilde{L_A} : \mathbb{C} \to \mathbb{C}$ defined by $\widetilde{L_A} = \varphi \circ L_A \circ \varphi^{-1}$ is $\mathbb{R}$-linear. Find conditions on a, b, c, d such that $\widetilde{L_A}$ is $\mathbb{C}$-linear. [2 + (1 + 5) = 8]

(2) Let U and W be subspaces of a finite dimensional vector spaces V such that $U + W = V$. Let

$$S := \{T \in \mathcal{L}(V) : T(U) \subseteq U \text{ and } T(W) \subseteq W\}.$$

(a) Prove that S is a subspace of the $\mathcal{L}(V)$.

(b) Suppose $\dim V = n, \dim U = a, \dim W = b$. Express the dimension of S in terms of n, a, b.

[2 + 5 = 7]

(3) Fix two matrices A and B in $M_{m\times n}(\mathbb{R})$. Define the linear map $T : M_{n\times m}(\mathbb{R}) \to M_{m\times n}(\mathbb{R})$ by $T(X) = AXB,$, $X \in M_{n\times m}(\mathbb{R})$. Prove that if $m \neq n$, then T is not invertible. [5]

(4) Prove or disprove: If f and g are two linear functionals on a finite dimensional vector space V over $\mathbb{C}$ such that $\ker f \subseteq \ker g$, then there exists $\alpha \in \mathbb{C}$ such that $g = \alpha f$. [5]

(5) Let $V = \mathbb{C}^3$, and define $\varphi_1, \varphi_2, \varphi_3 \in V'$ as follows:

$$\varphi_1(z_1, z_2, z_3) = z_1 - 2z_2,\ \varphi_2(z_1, z_2, z_3) = z_1 + z_2 + z_3,\ \varphi_3(z_1, z_2, z_3) = z_2 - 3z_3.$$

Show that $\{\varphi_1, \varphi_2, \varphi_3\}$ is a basis of V' by exhibiting a basis for V of which it is the dual basis. [5]

(6) Let A be a 3×3 matrix whose rank is 1. If its trace is 5, then what are the eigenvalues of A? Be sure to describe any multiplicities and explain your answer. [3]

(7) Let N be a 2×2 complex matrix such that $N^2 = 0$. Prove that either $N = 0$ or N is similar over $\mathbb{C}$ to

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

[5]

(8) Let $T : V \to V$ be a linear map and the dimension of V be n. Let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues of T with algebraic multiplicites $d_1, \ldots, d_k$. If β is an ordered basis for V such that $[T]_\beta$ is a upper triangular matrix, then prove that the diagonal entries of $[T]_\beta$ are $\lambda_1, \ldots, \lambda_k$ and that each λ_j occurs d_j times on the diagonal, $1 \leq j \leq k$. [4]

(9) Let $L_A : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by $L_A(x) = Ax$, $x \in \mathbb{R}^3$, where

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix}.$$

Show that L_A is diagonalizable and find an orthonormal basis $\mathcal{B}$ of $\mathbb{R}^3$ such that $[L_A]_\mathcal{B}$ is diagonal.

[3 + 5 = 8]

(10) Let $x_1, \ldots, x_n$ be n vectors in an inner product space V and A be a matrix whose ij-th entry is $\langle x_i, x_j \rangle$, $1 \leq i, j \leq n$. Show that $\det A = 0$ if and only if the vectors $x_1, \ldots, x_n$ are linearly dependent.

[5]

Good Luck!