PH3103 End-semester examination

Full marks : 50 **Time : 3 hrs**

Attempt 5 questions, taking at least 1 from each group

Group A

Q 1)[3+(1+3)+2] a) For what value(s) of α (if any) does the system of equations

$$x - 3y - z - 10t = \alpha, \qquad x + y + z = 5, \qquad 2x - 4t = 7, \qquad x + y + t = 4$$

have a solution? Find the general solution when it exists.

b) A 3×3 matrix J has eigenvalues of $0, \pm 1$.

(i) Show that $J^3 = J$

(ii) Express the matrix $\exp(-i\theta J)$ in terms of the 3×3 identity matrix, J and J^2.

c) Let A and B be $n \times n$ matrices with A invertible. Prove that

$$(A + B)\, A^{-1}\, (A - B) = (A - B)\, A^{-1}\, (A + B)$$

Q 2)[3+6] a) Let $\Delta(x_1, x_2, \ldots, x_8)$ be the following determinant

$$\Delta(x_1, \ldots, x_8) = \begin{vmatrix} x_1 & x_2 & x_6 & x_7 \\ x_3^2 & x_3 & x_2 & x_8 \\ x_3 x_4^2 & x_4^2 & x_4 & x_2 \\ x_3 x_4 x_5^2 & x_4 x_5^2 & x_5^2 & x_5 \end{vmatrix}$$

Find the partial derivatives $\dfrac{\partial \Delta}{\partial x_i}$ for $i = 6, 7, 8$.

b) Find a matrix P for which $P^{-1}AP$ is diagonal, given

$$A = \begin{pmatrix} -3 & -7 & 19 \\ -2 & -1 & 8 \\ -2 & -3 & 10 \end{pmatrix}$$

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Q 3)[(1+2+2+2)+2]a) By a magic matrix we mean a real square matrix in which all row sums, all column sums, and both diagonal sums are equal to some value σ.

i) If $M = [m_{ij}]$ is a 3×3 magic matrix, prove that $\sigma = 3m_{22}$

ii) Show that, given $a, b, c \in \mathbb{R}$ there is a unique 3×3 magic matrix $M(a, b, c)$ such that

$$m_{22} = a,\ m_{11} = a + b,\ m_{31} = a + c$$

iii) Show that the set $\{M(a, b, c) \,:\, a, b, c \in \mathbb{R}\}$ is a subspace of the vector space $\mathrm{Mat}_{3\times 3}[\mathbb{R}]$ of 3×3real matrices.

iv) Construct a basis of this vector subspace.

b) Let u, v and w form a basis of a vector space . Show that $u - v, u + 2w, u + v - 2w$ is also a basis for this vector space.

Group B

Q 4) [3+4+2] a) List and categorize the singularities of the following complex functions :

(i) $f_1(z) = \dfrac{\ln(z-2)}{(z^2 + 2z + 2)^4}$ **(ii)** $f_2(z) = \dfrac{\sin\sqrt{z}}{\sqrt{z}}$

b) Let the function

$$f\left(re^{i\theta}\right) = u(r, \theta) + iv(r, \theta)$$

be an analytic function of $z = re^{i\theta}$. Prove that

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

c) Show that both u and v above satisfy

$$\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r}\frac{\partial \Psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Psi}{\partial \theta^2} = 0$$

Q 5) [(2+1+1)+5]a) *(You* ***can not*** *use the residue theorem for this problem!)*

(i) If C is the circle $|z| = R$ prove that

$$\lim_{R\to\infty} \oint_C \frac{z^2 + 2z - 5}{(z^2+4)(z^2+2z+2)}dz = 0$$

(ii) Use this to deduce that the integral

$$\oint_{C_1} \frac{z^2+2z-5}{(z^2+4)(z^2+2z+2)}dz = 0$$

where C_1 is the circle $|z-2| = 5$.

(iii) Is the result in part (ii) true if C_1 is the circle $|z+1| = 2$? Explain.

b) Use the residue theorem to evaluate the integral

$$\int_0^{2\pi} \frac{e^{\cos\phi}\cos(\sin\phi)}{5-4\cos(\theta-\phi)}d\phi$$

Q 6) [4+5] a) Find the residue of $F(z) = \dfrac{\cot z \coth z}{z^3}$ at $z = 0$.

b) By the method of residues evaluate the integral

$$\int_0^{\infty} \frac{dx}{\sqrt[4]{x}\,(x+1)}$$

Group C

Q 7)[2+3+4] a) Let H be a subgroup of a group G. If $x \in yH$ then prove that $xH = yH$.

b) Let H and K be subgroups of a group G. If the orders of H and K are 56 and 63, respectively, prove that $H \cap K$ is cyclic.

c) Consider the group of symmetries of the (two-sided) equilateral triangle. Show that the rotations form a normal subgroup. Identify the quotient group.

Q 8)[4+5] a) The great orthogonality theorem states that

$$\sum_{g\in G}\left[D^{(\alpha)}(g)\right]_{ii'}\left[D^{(\beta)}(g^{-1})\right]_{j'j} = \frac{|G|}{d_\alpha}\delta^{\alpha\beta}\delta_{ij}\delta_{i'j'}$$

where the symbols have their usual meanings.

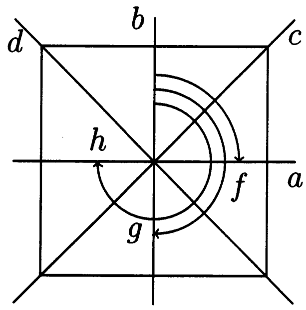
Using this show that the vectors with components equal to $\sqrt{\frac{N_C}{|G|}}\chi^{(\alpha)}(C)$, one for each class of G, are orthogonal to each other.

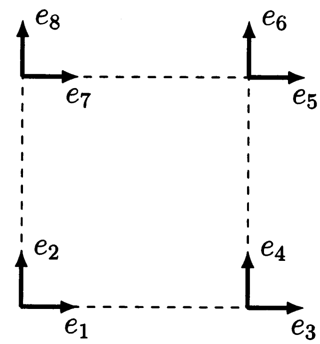
b) Construct the character table formed by the group of symmetries of the equilateral triangle. Remember - this group comprises of three rotations and three reflections.

Group D

Q 9)[2+2+3+2+5] Consider the molecular vibrations in a plane of the fictitious "squarene" molecule. The character table for its symmetries, the symmetry operations and the starting basis is shown below

	$\{e\}$	$\{a,b\}$	$\{c,d\}$	$\{f,h\}$	$\{g\}$
$\chi^{(1)}$	1	1	1	1	1
$\chi^{(2)}$	1	-1	1	-1	1
$\chi^{(3)}$	1	1	-1	-1	1
$\chi^{(4)}$	1	-1	-1	1	1
$\chi^{(5)}$	2	0	0	0	-2

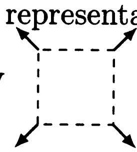
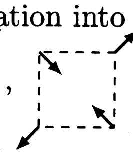
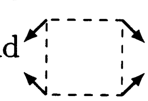




a) Show that the atomic displacements (in the plane) of the squarene molecule form a 8 dimensional representation with the characters

$$\chi(g) = \begin{cases} 8 & \text{for } g = e \\ 0 & \text{for } g \neq e \end{cases}$$

b) Use the character table to find the reduction of this representation into irreps.

c) Show that the displacements denoted schematically by  ,  and  form basis vectors for three 1-D irreducible representations.

Argue why these should correspond to normal modes of vibration.

d) By using the projection operator $\mathcal{P}_{nn}^{(\beta)} \equiv \sum_{g \in G} \left[D^{\beta}(g^{-1})\right]_{nn} L(g)$ determine the basis vectors for the two 2-dimensional invariant subspaces.

e) The normal frequencies of vibration can be found by solving the eigenvalue equation $\omega^2 x = \omega_0^2 \mathbb{F} x$ where $\mathbb{F}$ is the (scaled) force constant matrix. If the force constant matrix in the original $(e_1, e_2, \ldots, e_8)$ basis is given by

$$\mathbb{F} = \begin{pmatrix} 35 & 0 & -25 & -10 & 5 & 0 & -15 & 10 \\ 0 & 35 & 10 & -15 & 0 & 5 & -10 & -25 \\ -25 & 10 & 35 & 0 & -15 & -10 & 5 & 0 \\ -10 & -15 & 0 & 35 & 10 & -25 & 0 & 5 \\ 5 & 0 & -15 & 10 & 35 & 0 & -25 & -10 \\ 0 & 5 & -10 & -25 & 0 & 35 & 10 & -15 \\ -15 & -10 & 5 & 0 & -25 & 10 & 35 & 0 \\ 10 & -25 & 0 & 5 & -10 & -15 & 0 & 35 \end{pmatrix}$$

find the normal frequencies. *Hint : you already (almost) know the eigenvectors!*