Indian Institute of Science Education & Research

Mid Term Examination - MA3102 Algebra I

Date : 18^{th} September 2018

INSTRUCTIONS

This is a **closed-book** exam.

You have 1 hour.

- The examination is scored out of **20 points** and has **TWO** parts.
- You must do any **FOUR** problems from Part I and any ~~**TWO**~~ ONE problems from Part II.
- Each problem in part I (resp. part II) is worth 3 (resp. 8) points.
- Only the **BEST FOUR** answers from Part I will be counted even if you attempt all five.
- Only the **BEST** answer from Part II will be counted even if you attempt all two.
- There is a bonus problem worth 3 points. Only complete solutions will be awarded points.

Good luck!

PART I

Problem 1 Let $\sigma \in S_n$ be an element of order 2. Prove that σ is a product of disjoint transpositions. [*If you are using the fact that any permutation can be written as a product of disjoint cycles then you have to supply a proof.*]

Problem 2 Compute the centre of $GL_2(\mathbb{R})$.

Problem 3 Describe all group isomorphisms from $\mathbb{Z}_2 \times \mathbb{Z}_2$ to itself.

Problem 4 Determine whether the group $\mathbb{Z}_{12}^{\times}$ is cyclic or not.

Problem 5 Consider the group $\mathbb{R}/\mathbb{Q}$ and the operation

$$m : \mathbb{R}/\mathbb{Q} \times \mathbb{R}/\mathbb{Q} \longrightarrow \mathbb{R}/\mathbb{Q},\ m(x+\mathbb{Q}, y+\mathbb{Q}) = xy + \mathbb{Q}.$$

Prove or disprove: *The above binary operation is well defined.*

PART II

Problem A Let S_n be the symmetric group on n letters.
(i) Show that $\tau(i_1\, i_2 \cdots i_k)\tau^{-1} = (\tau(i_1)\, \tau(i_2) \cdots \tau(i_k))$ for any $\tau \in S_n$.
(ii) Find all elements $\tau \in S_6$ such that $\tau(1\,2)(3\,4)\tau^{-1} = (1\,2)(3\,4)$. [*Any answer without justification or logic will receive no credit.*]

Problem B Let G be a group of size 6.
(i) Show that there is an element x of order 2 and an element y of order 3.
(ii) Show that xyx^{-1} is some power of y. Hence, or otherwise, determine G up to isomorphism.

Bonus Problem Determine the group obtained in problem 3.