IISER Kolkata
End-Semester Examination
Fourth Year: Semester VII; 2018
PH4106 (Quantum Field Theory)
Time Three Hours; Full Marks 50
Answer all questions

1. a) Consider the action $S = \int \mathcal{L} d^4x$, where the Lagrangian density $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2$. If you now vary this action with respect to ϕ, show that the resulting Euler-Lagrange equation is $\Box\phi + m^2\phi = 0$, where the signature convention for the metric is $+, -, -, -$.

b) Consider the Dirac equation in a central potential. Show that there is indeed a problem with the conservation of orbital angular memementum.
Also show that the introduction of the helicity operator $\frac{1}{2}\hbar\mathbf{\Sigma}$, where $\mathbf{\Sigma} = diag(\sigma, \sigma)$, can lead to a conservation of "total angular momentum".

c) Show that ψ in Dirac equation

$i\hbar\frac{\partial\psi}{\partial t} + i\hbar c\alpha.\nabla\psi - \beta mc^2\psi = 0$

does have an interpretation as a probability amplitude.

2. The Lagrangian density $\mathcal{L}$ for a complex scalar field is given as

$$\mathcal{L} = \partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi,$$

where ϕ is a complex scalar field.

a)Show that the Lagrangian remains invariant under the transformation

$$\phi \to e^{-i\theta}\phi.$$

b)Find the expression for the conserved current
$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)}\delta\phi^\dagger$, and hence find an expression for the charge Q.

c) Given that Q, in normal order, can be written as $Q = \int d^3\mathbf{k}(a^\dagger_\mathbf{k} a_\mathbf{k} - b^\dagger_\mathbf{k} b_\mathbf{k})$, where $(a^\dagger_\mathbf{k}, a_\mathbf{k})$ and $(b^\dagger_\mathbf{k}, b_\mathbf{k})$ are pairs of annihilation and creation operators, show that $Q|k >= |k>$ and $Q|\tilde{k} >= -|\tilde{k}>$. $[a^\dagger_\mathbf{k}|0>= |k>$ and $b^\dagger_\mathbf{k}|0>= |\tilde{k}>]$

3. The Lagrangian density $\mathcal{L}$ for a Dirac field is given by

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu m)\psi.$$

a)Show that this Lagrangian is not Hermitian. Also show that another option

$$\mathcal{L}' = \frac{1}{2}\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \frac{1}{2}\bar{\psi}(i\gamma^\mu\overleftarrow{\partial_\mu} - m)\psi,$$

is Hermitian and differs from the former only by a total derivative term.

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b) Using $\mathcal{L}$, find out the conjugate momenta and construct the Hamiltonian density.

c) In terms of the annihilation and creation operators, the Hamiltonian, in normal order, can be written as $H = \Sigma_{s=\pm\frac{1}{2}}\int d^3\mathbf{k}E_k(c^\dagger(\mathbf{k}, s)c(\mathbf{k}, s) - d(\mathbf{k}, s)d^\dagger(\mathbf{k}, s))$.
Write down this Hamiltonian in the "normal order". Show that $H^{NO}|0>= 0$.

4. a) The Maxwell field is described by the Lagrangian density $\mathcal{L} = -\frac{1}{4\pi}F^{\mu\nu}F_{\mu\nu}$ where $F_{\mu\nu}$ is given in terms of the vector potential A^μ as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. With A_μ as the field variable, find out the conjugate momenta.

b) Find the corresponding Hamiltonian density.

c) With the Coulomb gauge $\nabla.\mathbf{A} = 0$, show that (using Maxwell,s equation) the vector potential components satisfy Klein-Gordon type equations for a massless particle.

d) Use the plane wave solutions $\mathbf{A} \sim \epsilon(\mathbf{k})e^{\pm ik_\mu x^\mu}$ to show that the polarization vector ϵ has only two independent components.