Subject: Intermediate Quantum Mechanics (PH3102)
End-Semester Examination

Number of students: 70

Answer any six questions.

1. (a) For a 1D quantum harmonic oscillator, obtain the matrix element x_{12} of the position operator $\hat{x}$ and the matrix element p_{21} of the momentum operator $\hat{p}$ in the energy eigenvector basis. (6)

   (b) Is a silver atom a fermion or a boson? The two stable isotopes are ^{107}Ag and ^{109}Ag, and silver has atomic number $Z = 47$. Justify your answer briefly. (1)

2. (a) For a 1D quantum harmonic oscillator, given the one-particle states and the energy eigenvalues, write down the two particle wavefunctions for i) distinguishable particles, ii) identical fermions iii) identical bosons. (3)

   (b) Write down the Hamiltonian for the two-particle system and show that the fermionic wavefunction found above is an eigenfunction of $\hat{H}$. Find the corresponding eigenvalues. (4)

3. (a) Consider a 1D two particle system whose wave function at $t = 0$ is

$$\psi(x_1, x_2) = \sqrt{\frac{1}{\pi a_1 a_2}} e^{-x_1^2/2a_1^2} e^{-x_2^2/2a_2^2}$$

   Find the expectation values of x_2, p_1, and $(x_1 - x_2)^2$. (4)

   (b) Evaluate the expectation value of S_z for a particle in the spin state $|\uparrow_n\rangle$, where the direction $\vec{n}$ is given by $\theta = 60°$ and $\phi = 0$. (3)

4. Consider a particle with spin $\frac{1}{2}$.

   (a) Find the eigenvalues and eigenfunctions of the operator $\widehat{S_x} + \widehat{S_y}$. (5)

   (b) Assume that $|\alpha\rangle$ designates the eigenfunction of $\widehat{S_x} + \widehat{S_y}$ that belongs to the maximal eigenvalue, and that the particle is in state $|\alpha\rangle$. If we measure the spin in the z direction, what are the values and probabilities? (2)

5. (a) A pair of spin-$\frac{1}{2}$ particles is produced in the singlet state, and are allowed to move in the $+y$ and $-y$ directions. The spins are measured with two Stern-Gerlach apparatuses SG1 and SG2 placed at some distance and oriented at 0° and 60° to the vertical, respectively. What are the probabilities that (i) both will record $+\frac{\hbar}{2}$, (ii) both will record $-\frac{\hbar}{2}$, (iii) one will record $+\frac{\hbar}{2}$ and the other $-\frac{\hbar}{2}$? (3)

   (b) A particle is placed in the 1D harmonic oscillator potential $V(x) = 2x^2$. Its position space has been discretized into 5 segments $x = [-2 \quad -1 \quad 0 \quad 1 \quad 2]$ for the purpose of running a simulation. Write down the matrices corresponding to the potential and kinetic energy operators using the finite difference formula

$$\frac{d^2u(x)}{dx^2} = \frac{u(x + \Delta x) - 2u(x) + u(x - \Delta x)}{\Delta x^2}$$

(4)

6. (a) Consider a particle in the 2D symmetrical infinite potential well, which is subjected to perturbation $W = Cxy$, where C is a constant. Calculate the 1st-order corrections to the ground state energy of the system. (3)

(b) For a harmonic oscillator potential with force constant k and reduced mass m, consider the perturbation $W = ax^3$. First state the condition(s) under which perturbation theory may be applied and then calculate the 1st order corrections to the ground state energy. (2)

(c) For a 1D infinite potential well of width a, calculate the 1st order corrections to the ground state energy for the perturbation (2)

$$W(x) = a\omega_0\delta(x - a/2)$$

7. (a) The spherical harmonics are given by

$$Y_l^m(\theta,\phi) = \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}P_l^m(\cos\theta)e^{im\phi}$$

where $P_m^l(x)$ are associated legendre polynomials.

$$P_l^m(x) = (-1)^m(1-x^2)^{\frac{|m|}{2}}\frac{d^{|m|}}{dx^{|m|}}P_l(x)$$

and

$$P_l(x) = \frac{1}{2^l\, l!}\left(\frac{d}{dx}\right)^l(x^2-1)^l$$

Compute all the possible eigenfunctions $Y_1^m(\theta,\phi)$ (5)

(b) Explain, in terms of quantum mechanics, why a block of iron is hard, i.e., cannot be compressed. (2)

Things you may have forgotten:

- For a quantum harmonic oscillator, the first two energy eigenfunctions are

$$\psi_0(x) = \left(\frac{1}{\sqrt{\pi}a}\right)^{1/2}e^{-x^2/2a^2}, \qquad \psi_1(x) = \left(\frac{1}{2\sqrt{\pi}a}\right)^{1/2}\frac{2x}{a}e^{-x^2/2a^2},$$

where $a = \sqrt{\frac{\hbar}{m\omega}}$ is the characteristic length parameter of the oscillator.

- In a general direction $\vec{n}$, the up- and down-components of spin are given by

$$|\uparrow_n\rangle = \begin{bmatrix}\cos\theta/2\\ e^{i\phi}\sin\theta/2\end{bmatrix}, \qquad |\downarrow_n\rangle = \begin{bmatrix}-e^{-i\phi}\sin\theta/2\\ \cos\theta/2\end{bmatrix}$$

and the operator is

$$\widehat{S}_n = \frac{\hbar}{2}\begin{bmatrix}\cos\theta & e^{i\phi}\sin\theta\\ e^{i\phi}\sin\theta & -\cos\theta\end{bmatrix}$$

- $\int_{-\infty}^{\infty}x^{2n}\,e^{-x^2}\,dx = \frac{1\times3\times\cdots\times(2n-1)}{2^n}\sqrt{\pi},\ n\geq1;\quad \int_{-\infty}^{\infty}x^{2n+1}\,e^{-x^2}\,dx = \frac{n!}{2},\ n\geq0$

Fill in the correct bubbles. Marks will not be awarded without appropriate rough work.

Name: ______________________________

Roll No: ______________________________

1. $[\widehat{H}, \widehat{Q}] = 0$ implies (1)

   ○ Eigenfunctions of $\widehat{Q}$ are stationary ○ $\widehat{Q}$ is a conserved quantity
   ○ Both of the above ○ None of the above

2. Which of the following statement(s) is(are) always true about any quantum state $|\psi\rangle$? (1)

   ○ $H|\psi\rangle$ gives the result of an energy measurement on the state $|\psi\rangle$.
   ○ Immediately after energy measurement, the system is in the state $H|\psi\rangle$.
   ○ There exists a definite energy E associated with the state $|\psi\rangle$ given by $H|\psi\rangle = E|\psi\rangle$
   ○ None of the above

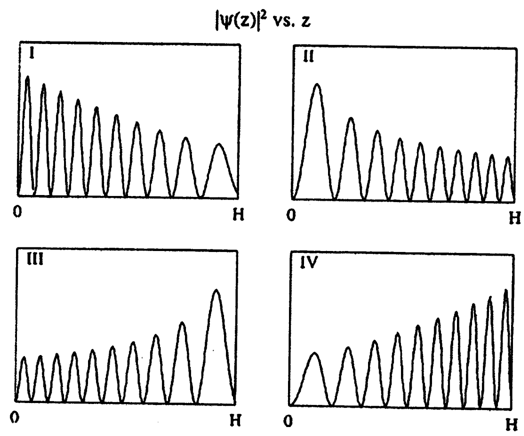
3. A particle is prepared in a simultaneous eigenstate of $\mathbf{L}^2$ and L_z. If $l(l+1)\hbar^2$ and $m\hbar$ are respective eigenvalues of $\mathbf{L}^2$ and L_z then the expectation value of $\langle L_x^2\rangle$ of the particle in this state satisfies (1.5)

   ○ $\langle L_x^2\rangle = 0$ ○ $0 \leq \langle L_x^2\rangle \leq l^2\hbar^2$ ○ $0 \leq \langle L_x^2\rangle \leq \frac{l(l+1)\hbar^2}{3}$ ○ $\frac{l\hbar^2}{2} \leq \langle L_x^2\rangle \leq \frac{l(l+1)\hbar^2}{2}$

4. In a basis in which the z component S_z of the spin is represeted by a diagonal matrix, an electron is in a spin state $\psi = \begin{bmatrix} \frac{1+i}{\sqrt{6}} \\ \sqrt{\frac{2}{3}} \end{bmatrix}$. The probabilities that the measurement of S_z will yield the values $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ are, respectively, (1.5)

   ○ $\frac{1}{2}$ and $\frac{1}{2}$ ○ $\frac{2}{3}$ and $\frac{1}{3}$ ○ $\frac{1}{4}$ and $\frac{3}{4}$ ○ $\frac{1}{3}$ and $\frac{2}{3}$

5. A particle is dropped from a height H under the influence of gravity and bounces, without loss of energy, from a flat surface (at $z = 0$) back up to the same height. Which of the plots below best represents the quantum mechanical position-space probability density $|\psi(z)|^2$ vs z, of an energy eigenstate for this system? (1.5)



   ○ I ○ II ○ III ○ IV ○ None of them are possible solutions.

6. For a particle in a one-dimensional infinite square well, which state will have the fastest variation in time for the position probability density? (The state ψ_n corresponds to an energy of E_n) (1.5)

   ○ ψ_4 ○ ψ_1 ○ $\frac{\psi_2+\psi_3}{\sqrt{2}}$ ○ $\frac{\psi_1-\psi_3}{\sqrt{2}}$