Department of Mathematics and Statistics, IISER Kolkata
Analysis-III (MA3101)
End Semester Examination, Total Marks: 50, Time: 3 hrs.

Instructor: Shirshendu Chowdhury   Date: 5th December, 2018.

Question 1. [$2 \times 5 = 10$]

Justify True or False. If it is true prove it other wise give counter example or proper reasons.

(i) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by

$$f(0,0) = 0 \text{ and } f(x,y) = \frac{x|y|}{\sqrt{x^2+y^2}}, \ (x,y) \neq (0,0)$$

Then f is differentiable at $(0,0)$.

(ii) The origin $(0,0) \in \mathbb{R}^2$ is the only critical point of $f(x,y) = x^2 + y^2(1-x)^3$ where f attains its local minimum, but there is no global minimum.

(iii) Let $a, x \in \mathbb{R}^n$ and $a \neq 0$. Then

$$\frac{1}{||a||^2} = \min\{||x||^2 : (a,x)_{\mathbb{R}^n} = 1\}.$$

(iv)

$$\oint_C (x^2dx + y^2dy) \neq 0$$

for the closed path given by the ellipse $4x^2 + 9y^2 = 36$.

(v) $S_1 := \{(x,y) \in \mathbb{R}^2 | (x-1)^2 + y^2 = 1\}$ and $S_2 := \{(x,y) \in \mathbb{R}^2 | (x+1)^2 + y^2 < 1\}$ are not path connected.

Question 2. [5+5=10]

(i) Let

$$B = \{x \in \mathbb{R}^n : ||x|| < 1\} \text{ and } \bar{B} = \{x \in \mathbb{R}^n : ||x|| \leq 1\}.$$

Let $f : \bar{B} \longrightarrow \mathbb{R}$ be continuous on $\bar{B}$ and differentiable on B. Show that if $f(x) =$ constant on $\partial B = \{x \in \mathbb{R}^n : ||x|| = 1\}$, then there exists a $c \in B$ such that $\nabla f(c) = 0$.

(ii) Let $f : \Omega \longrightarrow \mathbb{R}$ be twice differentiable on the open set $\Omega \subset \mathbb{R}^n$ and

$$S = \{x \in \Omega : \nabla f(x) = 0; \ \det[D^2 f(x)] \neq 0\}.$$

Prove that points of S are isolated i.e. show that for $x \in S$, there exists a $\delta > 0$ such that for any $0 < ||h|| < \delta$, $\nabla f(x+h) \neq 0$.

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5/12/2018

Question 3. [2+5+5=12]

(i) ~~Sate~~ State Inverse Function Theorem.

(ii) Let $f : \mathbb{R}^3 \longmapsto \mathbb{R}^3$ be

$$(u, v, w) = f(x, y, z) = (x - xy, xy - xyz, xyz) \text{ and } U = \{(x, y, z) : xy \neq 0\}.$$

Show that f is invertible on U and determine a formula for the inverse $f^{-1} : f(U) \longmapsto U$. Find the Jacobian of f^{-1} at a point (a, b, c) of $f(U)$.

(iii) Show that there are points which satisfy the equations

$$x - e^u \cos v = 0, \quad y - e^u \sin v = 0.$$

Let $A = (x_0, y_0, u_0, v_0)$ be such a point. Show that in a neighbourhood of A there exists a unique solution $(u, v) = f(x, y)$. Prove that $\det[Df(x, y)] = \frac{y}{x}$.

Question 4. [5+5=10]

(i) Show that the function

$$f(x, y, z) = (x + y + z)^3 - 3(x + y + z) - 24xyz + 8$$

has a maximum at $(-1, -1, -1)$ and minimum at $(1, 1, 1)$.

(ii) Let $f(x, y) = 3x^2 - 2y^2 + 2y$. Find the maximum and minimum values of f on the set

$$B = \{(x, y) : x^2 + y^2 \leq 1\}.$$

Question 5. [4+2+2=8]

(i)Let F in $\mathbb{R}^3$ be given by

$$F(x, y, z) = (y^2 \cos x + z^3, 2y \sin x - 4, 3xz^2 + 2).$$

Determine whether or not F is the gradient of a scalar field. When F is a gradient, find a corresponding potential function φ.

(ii) Let F in $\mathbb{R}^2$ be given by

$$F(x, y) = (x + y, x - y).$$

Show that the line integral of F along a curve

$$\alpha(t) = (f(t), g(t)) \quad a \leq t \leq b$$

depends only on $f(a), f(b), g(a), g(b)$, where $f, g : [a, b] \to \mathbb{R}$ are piecewise smooth.

(iii) Prove that the set $\mathbb{R} \setminus \{0\}$ with the usual Euclidean topology is disconnected.