INDIAN INSTITUTE OF SCIENCE EDUCATION & RESEARCH KOLKATA

Mathematical Statistics - I (ID4105) – Final Exam

Date: 27th November, 2018 Duration: 3 hours

Maximum points that you can score is 100. Good luck!

Question 1 (10 points)

Consider a sequence of random variables $\{X_n : n \geq 1\}$, where $X_n \sim \mathrm{N}(0, n^{-\alpha})$ for each n and $\alpha > 0$ is a fixed number. Show that X_n converges to zero *almost surely*.

[*Hint: For a random variable $Y \sim \mathrm{N}(0, \sigma^2)$ and any integer $m \geq 1$, $\mathbb{E}(Y^{2m}) = c_m \sigma^{2m}$, where $c_m > 0$. Use the fact $\mathbb{P}(|Y| > \epsilon) = \mathbb{P}(Y^{2m} > \epsilon^{2m})$ along with Markov's inequality.*]

Question 2 (30 points)

Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \mathrm{Unif}(\theta, 1)$, where $\theta \in (-\infty, 1)$. We want to test the hypothesis that $H_0 : \theta = 0$ versus $H_1 : \theta < 0$. For this purpose, we wish to use the test

$$\text{Reject } H_0 \text{ if } X_{(1)} < k.$$

(a) Determine k such that the level of the above test will be exactly equal to $\alpha \in (0, 1)$. [4]

(b) Calculate the power function of the test in part (a). [8]

(c) Derive the most powerful test for $H_0 : \theta = 0$ versus $H_1 : \theta = \theta_1$, where $\theta_1 < 0$. Denote the test function by $\delta(\mathbf{X})$. [8]

(d) Is the test derived in part (a) the uniformly most powerful (UMP) test for $H_0 : \theta = 0$ versus $H_1 : \theta < 0$? Justify your answer. Is the same conclusion true for the test derived in part (c)? If yes, why? If no, can you slightly modify the region $[\delta(\mathbf{X}) = 1]$ so that the resulting test becomes UMP? [10]

[*Hint: For the last part, look carefully at the term $\mathbb{P}_{\theta_1}[\delta(\mathbf{X}) = 1]$ when calculating the power.*]

Question 3 (30 points)

Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \mathrm{N}(\theta, 1)$, where $\theta \in \mathbb{R}$ is an unknown parameter. We want to estimate $\Phi(\theta)$, where Φ is the cdf of $\mathrm{N}(0, 1)$. Denote $\mathbf{X} = (X_1, X_2, \ldots, X_n)^T$.

(a) Show that $\delta(\mathbf{X}) := \mathbf{1}(X_1 > 0)$ is an unbiased estimator of $\Phi(\theta)$. [2]

(b) Show that the joint distribution of X_1 and $\overline{X} := n^{-1} \sum_{i=1}^{n} X_i$ is a bivariate Normal distribution. Find its parameters. [6]

(c) Use the joint distribution obtained in part (b) to find $T(\mathbf{X}) = \mathbb{E}[\delta(\mathbf{X}) \mid \overline{X}]$ in terms of Φ. [5]

(d) Find $\mathbb{E}_\theta[T(\mathbf{X})]$. Suppose that $\eta(\mathbf{X})$ is some other estimator of $\Phi(\theta)$ (different from $T(\mathbf{X})$) such that

Aira Chy 27.11.18

$\mathbb{E}_\theta[\eta(\mathbf{X})] = \Phi(\theta)$ for all θ. Show that $\mathrm{Var}_\theta(T(\mathbf{X})) \leq \mathrm{Var}_\theta(\eta(\mathbf{X}))$ for all θ. Can equality hold for all θ? [7]

(e) Find the MLE, say S_n, of $\phi(\theta)$. Find the variance of the limiting distribution of $\sqrt{n}\{S_n - \phi(\theta)\}$. [6]

(f) Suppose that we want to estimate $\phi(\theta)$, where ϕ is the pdf of N(0, 1), i.e., $\phi = \Phi'$. It is known that $V_n := a_n\phi\left(a_n\overline{X}\right)$ is the UMVUE of $\phi(\theta)$, where $a_n = \sqrt{n/(n-1)}$. Use part (e) to show that

$$\mathrm{Var}_\theta(V_n) \geq \frac{\theta^2\phi^2(\theta)}{n} \qquad \forall\, n \geq 1 \quad \& \quad \forall\, \theta \in \mathbb{R}. \qquad [4]$$

[*Hint: Simplify* $\phi'(\theta)$.]

Question 4 (30 points)

Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} f_\lambda(x) = \lambda^{-1}\exp(-x/\lambda)\mathbf{1}(x > 0)$, where $\lambda > 0$ is an unknown parameter. We want to test $H_0 : \lambda = \lambda_0$ versus $H_1 : \lambda \neq \lambda_0$.

(a) Show that the likelihood ratio test (LRT) for the above problem rejects the null hypothesis if $\overline{X} > c_1$ OR $\overline{X} < c_2$, where $\overline{X} = n^{-1}\sum_{i=1}^n X_i$, and c_1, c_2 are some constants. [6]

(b) Without using Wilks' theorem, find one set of c_1 and c_2 such that the level of the LRT is asymptotically (i.e., as $n \to \infty$) equal to $\alpha \in (0, 1)$. [7]

[*Hint: You may use the facts that* $\mathbb{E}_\lambda[X] = \lambda$ *and* $\mathrm{Var}_\lambda(X) = \lambda^2$.]

(c) Show that the power of the test that you derived in part (b) converges to 1 as $n \to \infty$ for any fixed $\lambda \neq \lambda_0$. [5]

(d) Under the alternative hypothesis, consider the sequence of values of λ given by $\lambda_n = \lambda_0 + n^{-1/2}$, i.e., the "alternatives shrink towards the null". Let β_n be the power of the test when $\lambda = \lambda_n$. Show that β_n converges to some value $\beta > \alpha$ as $n \to \infty$. Do the same calculation for $\lambda_n = \lambda_0 + n^{-3/4}$ and $\lambda_n = \lambda_0 + n^{-1/4}$. What do you think is the reason for the behaviour that you observe in these three situations? [12]

[*Hint: You may use the fact that under all of the above sequence of alternatives,* $\sqrt{n}(\overline{X} - \lambda_n)/\lambda_n$ *converges in distribution to* $N(0, 1)$ *as* $n \to \infty$.]

Question 5 (10 points)

Suppose that X and Y are two random variables with finite variances such that

(i) $\mathbb{E}[X] = \mathbb{E}[Y]$, and

(ii) for some $\beta \neq 0$ and all $x, y \in \mathbb{R}$,

$$\mathbb{E}[X \mid Y = y] = \beta y \qquad \text{and} \qquad \mathbb{E}[Y \mid X = x] = \frac{x}{\beta}.$$

Show that $\mathbb{P}(X = Y) = 1$.

[*Hint: Find the value of* β. *Then, find* $\mathbb{E}[(X - Y)^2]$ *by using conditional expectation on* X *as well as on* Y.]

– END OF THE EXAM –

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