IISER Kolkata **Final Semester Examination**

MA 4102 : Functional Analysis

Date : December 4, 2018 **Time : 10h00 - 13h00**

Problem 1.
Let X, Y be normed $\mathbb{K}$-linear spaces and let $T : X \to Y$ be $\mathbb{K}$-linear. Show that $T \in \mathcal{B}(X;Y)$ if and only if $T : (X, \mathcal{T}_w^X) \to (Y, \mathcal{T}_w^Y)$ is continuous. [6 points]

Problem 2.
Let X be a Banach space over $\mathbb{K}$. Prove that the weak* topology on X^* is metrizable if and only if X is finite dimensional. [7 points]

Problem 3.
Let $(X, \|\cdot\|_1)$ be a reflexive normed $\mathbb{K}$-linear space and let $\|\cdot\|_2$ be a norm on X that is equivalent to $\|\cdot\|_1$. Prove that $(X, \|\cdot\|_2)$ is reflexive. [6 points]

Problem 4.
Prove the following

A. $L^\infty(0,1)$ is not separable. [3 points]

B. $L^1(0,1)$ is not reflexive. [3 points]

C. $L^\infty(0,1)$ is not reflexive. [3 points]

D. $C[0,1]$ is not reflexive. [5 points]

Problem 5.
Let X be a normed $\mathbb{K}$-linear space and let M be a closed subspace of X. Show that $M^\perp$ is isometrically isomorphic to $(X/M)^*$. [9 points]

Problem 6.
Let $1 \leqslant p < q \leqslant \infty$ and let $\iota : L^q(0,1) \to L^p(0,1)$ be the inclusion map. Is ι compact? [6 points]

Problem 7.
Let X be a compact metric space, let μ be a Radon outer measure on X and let $k \in C(X \times X; \mathbb{R})$. Let us define $I : C(X) \to C(X)$ as

$$I(f)(x) := \int_X k(x,y) f(y)\, d\mu(y), \text{ for all } f \in C(X),\ x \in X.$$

Prove that I is compact. [6 points]

Problem 8.
Let X, Y be Banach spaces and let $T \in \mathcal{B}(X;Y)$ be compact. Prove that T is of finite rank if and only if $\operatorname{im} T$ is closed in Y. [6 points]

Problem 9.
Let X be a reflexive normed $\mathbb{K}$-linear space and let $f \in X^*$. Show that, there exists a $x_0 \in X$ such that $\|x_0\| = 1$ and $f(x_0) = \|f\|$. [5 points]