PH4105/5105 Mid-semester examination

Full marks : 20 **Time : 60 mins**

Attempt all questions

Q 1) a) Prove that if $d : M \times M \to \mathbb{R}$ is a metric on M, then $\overline{d}$ defined by

$$\forall x, y \in M, \qquad \overline{d}(x,y) = \frac{d(x,y)}{1 + d(x,y)}$$

is also a metric on M.

b) Show that every open ball according to metric d is also an open ball according to $\overline{d}$ [3 + 2]

Q 2) Let $\psi : M \to N$ be a smooth map between differentiable manifolds M and N.

a) Define the pullback ψ^* that acts on $C^\infty(N)$ to give elements of $C^\infty(M)$.

b) Prove that the map ψ^* is linear.

c) Show that $\psi^*(fg) = (\psi^* f)(\psi^* g)$

d) Let $v_p \in T_pM$ for some $p \in M$. Show that the object defined by

$$x[f] = v_p[\psi^* f] \qquad \forall f \in C^\infty(N)$$

is a tangent vector to N at the point $\psi(p)$. [1 + 1 + 1 + 2]

Q 3) a) Show that for $X, Y \in \mathfrak{X}(M)$, $[X, Y] \in \mathfrak{X}(M)$

b) Show that for $X, Y \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$ we have

$$[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X$$

c) On $\mathbb{R}^2$calculate $[X, Y]$ for

$$X = x\left(\frac{\partial}{\partial x}\right) - y\left(\frac{\partial}{\partial y}\right), \qquad Y = y\left(\frac{\partial}{\partial x}\right) + x\left(\frac{\partial}{\partial y}\right)$$

[2 + 2 + 1]

Q 4) a) Show that

$$\phi\left((x,y),t\right) = \left(e^{at}x, e^{bt}y\right)$$

is a one parameter group of transformations on $\mathbb{R}^2$ and find its infinitesimal generator.

b) Show that if ϕ is a one parameter group of transformations on M, and X is its infinitesimal generator then the curve ϕ_x is an integral curve of X that starts at X. [3 + 2]