MA 4102 : Functional Analysis

Date : November 2, 2018 **Time : 17h00 - 18h00**

Problem 1.
Let X be a normed $\mathbb{K}$-linear space and let $E \subsetneq X$ be a closed, proper subspace. Calculate

$$\sup_{x \in X \setminus \{0\}} \frac{d(x, E)}{\|x\|}.$$

[5 points]

Problem 2.
Let $1 \leqslant p \leqslant \infty$. We consider $\mathbb{R}^2$ endowed with the $\|\cdot\|_p$-norm. Let $M := \{(x, 0) : x \in \mathbb{R}\}$ and let $f : M \to \mathbb{R}$ be defined as

$$f(x, 0) := x, \text{ for all } x \in \mathbb{R}.$$

Find all Hahn-Banach extensions of f. [5 points]

Problem 3.
Let X be a normed $\mathbb{K}$-linear space and let $f \in X^* \setminus \{0\}$. Prove that f is an open map. [5 points]

Problem 4.
Let X be a $\mathbb{R}$-linear space, let M be a subspace, let $\varphi : X \to [0, \infty)$ be convex and let $f : M \to \mathbb{R}$ be $\mathbb{R}$-linear satisfying

$$f(x) \leqslant \varphi(x), \text{ for all } x \in M.$$

Then, there exists a $\mathbb{R}$-linear functional $\bar{f} : X \to \mathbb{R}$ such that

(i) $\bar{f}(x) = f(x)$, for all $x \in M$, and
(ii) $\bar{f}(x) \leqslant \varphi(x)$, for all $x \in X$.

[6 points]

Hint : Consider the map $p : X \to [0, \infty)$ defined as

$$p(x) := \inf_{t>0} \frac{\varphi(tx)}{t}, \text{ for all } x \in X.$$