Indian Institute of Science Education and Research Kolkata
Department of Mathematics and Statistics
Numerical Analysis (MA3105)

Autumn Sem
December 3, 2018

Time: 10:00–13:00 | *End-Semester Exam* | Marks: 50

1. (i) State QR factorisation. [2]

   (ii) Find the QR factorisation for the following matrix

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

[5]

   (iii) Let $\|.\|$ be any norm on $\mathbb{R}^n$. Assume that maximum and minimum of the set

$$\{\|x\| : \|x\|_\infty = 1\}$$

exist. Show that there exist two positive constants $c_1$ and $c_2$ such that

$$c_1 \|x\|_\infty \le \|x\| \le c_2 \|x\|_\infty$$

for all $x \in \mathbb{R}^n$. Hence conclude that there exist two positive constants $m_1$ and $m_2$ such that

$$m_1 \|x\|_a \le \|x\|_b \le m_2 \|x\|_a ,$$

where $\|.\|_a$ and $\|.\|_b$ are any two norms on $\mathbb{R}^n$. [7]

   (iii) Let $A$ be any matrix of order $n$. Show that $\|A\|_2 = \|A^T\|_2$. Is it always true that $\|A\|_\infty = \|A^T\|_\infty$? [6]

2. (i) Let $x_0, x_1, \dots, x_n$ be distinct real numbers and $y_0, y_1, \dots, y_n$ be any real numbers. Let $c_0, c_1, \dots, c_n$ be real numbers such that

$$f(x) = \sum_{k=0}^{n} c_k e^{kx} \quad x \in \mathbb{R},$$

satisfies

$$f(x_k) = y_k \quad k = 0, 1, \dots, n.$$

Show that $c_0, c_1, \dots, c_n$ are unique. [6]



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(ii) Let $x_0, x_1, x_2$ be distinct points and suppose $f''$ is continuous in an interval containing them. Show that

$$f[x_0, x_1, x_2] = \int_0^1 \int_0^{t_1} f''\Big(x_0 + t_1(x_1 - x_0) + t_2(x_2 - x_1)\Big) dt_2 dt_1.$$

[8]

3. Let $Y : [x_0, x_M] \to \mathbb{R}$ be a solution of

$$(1) \qquad \begin{cases} \frac{dy}{dx} = f(x, y), \\ y(x_0) = y_0, \end{cases}$$

where $f$ is uniformly Lipschitz in $y$, that is, there exists a $L > 0$ such that

$$|f(x, y_1) - f(x, y_2)| \leq L\,|y_1 - y_2|$$

for all $x \in [x_0, x_M]$ and for all $y_1, y_2$. Assume that $Y \in C^2([x_0, x_M])$. Let $N \in \mathbb{N}$ and $h = (x_M - x_0)/N$. We define $x_n = x_0 + hn$ for $n = 0, 1, 2, \ldots, N$.

a. Let $y_0, y_1, \ldots, y_N$ be the solution of Euler's scheme corresponding to (1). Define the function $Y_N : [x_0, x_M] \to \mathbb{R}$ by

$$Y_N(x) = \frac{x_i - x}{h} y_{i-1} + \frac{x - x_{i-1}}{h} y_i \quad x \in [x_{i-1}, x_i], i = 1, 2, \ldots, N.$$

Using the error estimate of Euler's scheme, show that the sequence of functions $(Y_N)_{N \in \mathbb{N}}$ uniformly converges to $Y$ as $N \to \infty$. [6]

b. Consider the trapezium rule method corresponding to (1) as

$$y_{n+1} = y_n + \frac{h}{2}\Big(f(x_n, y_n) + f(x_{n+1}, y_{n+1})\Big).$$

for $n = 0, 1, \ldots, N - 1$.

(i) Assume that $\Phi(x, y; h)$ satisfies the following relation

$$\Phi(x, y; h) = \frac{1}{2}\Big(f(x, y) + f(x + h, y + h\Phi(x, y; h))\Big)$$

for $x \in [x_0, x_M]$, $y \in \mathbb{R}$ and $h > 0$. Show that there exists a positive $h_1$ such that $\Phi : [x_0, x_M] \times \mathbb{R} \times [0, h_1] \to \mathbb{R}$ is uniformly Lipschitz in $y$. [6]

(ii). Define the truncation error $T_n$ corresponding to above trapezium rule method. Show that under suitable condition on solution $Y$, there exists a positive constant $C$ (independent of $n$) such that $|T_n| \leq Ch^2$. [4]