MA5104: End Sem Exam

Full Marks: 50

29.11.2018

1. Let $\{X_t\}_{t=1}^{\infty}$ be a time series with finite second moments. Define

$$\widehat{X}_n = \begin{cases} 0, & \text{if } n = 1 \\ P_{n-1}X_n, & \text{otherwise,} \end{cases}$$

where $P_{n-1}X_n$ is the best linear predictor of X_n based on $X_1, \ldots, X_{n-1}$.

(a) Write Innovation $\mathbf{Iv}_n$ as $A\mathbf{X}_n$, where A is a lower triangular matrix and $\mathbf{X}_n = (X_1, \ldots, X_n)'$. [3]

(b) Write $\widehat{X}_n$ interms of $\mathbf{Iv}_n$. [2]

(c) Using second part, prove that

$$\widehat{X}_{n+1} = \begin{cases} 0, & \text{if } n = 0, \\ \sum_{j=1}^{n} \theta_{n,j}(X_{n+1,j} - \widehat{X}_{n+1,j}), & \text{otherwise,} \end{cases}$$

for appropriate θ's. [2]

(d) Prove that, for $0 \leq k < n$,

$$\text{cov}(\widehat{X}_{n+1}, X_{k+1} - \widehat{X}_{k+1}) = \text{cov}(X_{n+1}, X_{k+1} - \widehat{X}_{k+1}).$$

[3]

2(a) State explicitely all the assumptions (other than normality) of a general linear state space model given by

$$\begin{aligned} \mathbf{Y}_t &= G_t\boldsymbol{\alpha}_t + \mathbf{W}_t \\ \boldsymbol{\alpha}_{t+1} &= F_t\boldsymbol{\alpha}_t + \mathbf{V}_t, \end{aligned}$$

for $t = 1, 2, \ldots$. [3]

(b) Show that $\mathbf{V}_t$ is uncorrelated with $\boldsymbol{\alpha}_s$ and $\mathbf{Y}_s$, for $1 \leq s \leq t$. [2]

(c) Show that $\mathbf{W}_t$ is uncorrelated with $\mathbf{Y}_s$, for $0 \leq s < t$.

(d) Consider a MA(2) process as

$$Y_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2},$$

where $Z_t \sim \text{WN}(0, \sigma^2)$. Represent the MA(2) process as a state-space model. [3]

3. Recall in Kalman prediction we have the notation $P_{t-1}(\boldsymbol{\alpha}_t) := \widehat{\boldsymbol{\alpha}}_t$ as the best linear predictor of $\boldsymbol{\alpha}_t$ based on observations $\mathbf{Y}_1, \ldots, \mathbf{Y}_{t-1}$.

   (a) Using the general form of linear state space model prove that $P_{t-1}(\mathbf{Y}_t) = G_t\widehat{\boldsymbol{\alpha}}_t$.

   (b) Using the fact that $P_t(\cdot) = P_{t-1}(\cdot) + P(\cdot|\mathbf{I}_t)$, where $\mathbf{I}_t = \mathbf{Y}_t - G_t\widehat{\boldsymbol{\alpha}}_t$. Prove that

$$\widehat{\boldsymbol{\alpha}}_{t+1} = F_t\widehat{\boldsymbol{\alpha}}_t + E(\boldsymbol{\alpha}_{t+1}\mathbf{I}_t')[E(\mathbf{I}_t\mathbf{I}_t')]^{-1}\mathbf{I}_t.$$

[3]

   (c) Using the final expression of $\widehat{\boldsymbol{\alpha}}_{t+1}$ as $\widehat{\boldsymbol{\alpha}}_{t+1} = F_t\widehat{\boldsymbol{\alpha}}_t + \Theta_t\Delta_t^{-1}\mathbf{I}_t$ prove that

$$\Sigma_{t+1} = F_t\Sigma_tF_t' + Q_t - \Theta_t\Delta_t^{-1}\Theta_t'.$$

   where $\Sigma_t = E[(\boldsymbol{\alpha}_t - \widehat{\boldsymbol{\alpha}}_t)(\boldsymbol{\alpha}_t - \widehat{\boldsymbol{\alpha}}_t)']$, $E(\mathbf{V}_t\mathbf{V}_t') = Q_t$ and $E(\mathbf{I}_t\mathbf{I}_t') = \Theta_t$. [2]

   (d) Consider a simple model

$$Y_t = \alpha_t + W_t, W_t \sim \text{WN}(0, \sigma_w^2)$$
$$\alpha_{t+1} = \alpha_t + V_t, V_t \sim \text{WN}(0, \sigma_v^2)$$

   Show that $\hat{Y}_{t+1} = (1 - a_t)\hat{Y}_t + a_tY_t$, where $a_t = \frac{\Sigma_t}{\Sigma_t + \sigma_w^2}$. If Σ_t is free of t then find an closed form expression for Σ in terms of σ_v^2 and σ_w^2. [3]

4(a) Establish whether or not the following function is the acvf of a stationary process, using the spectral analysis,

$$\gamma(h) = \begin{cases} 1 & \text{if } h = 0, \\ -0.5 & \text{if } h = \pm 2, \\ -0.25 & \text{if } h = \pm 3, \\ 0, & \text{otherwise.} \end{cases}$$

[3]

   (b) Determine the acvf of the process whose spectral density is given by

$$f(w) = \frac{1 - 2|w|}{\pi}, -0.5 \leq w \leq 0.5.$$

[3]

   (c) Let a stationary process $\{X_t\}$ be given as $X_t = Z_t + 0.5Z_{t-1}$. Show that the spectral density of the process is

$$f(w) = \sigma^2[1.25 + \cos(2\pi w)],$$

   where $Z_t \sim \text{WN}(0, \sigma^2)$. [2]

   (d) For any j and k prove that

$$\sum_{t=1}^{n} \cos(2\pi tj/n)\sin(2\pi kj/n) = 0.$$

[2]

5(a) Let $\{X_t\}$ and $\{Y_t\}$ be uncorrelated stationary process with spectral distribution functions $F_X(\cdot)$ and $F_Y(\cdot)$. Show that the process $Z_t := X_t + Y_t$ is stationary and determine its spectral distribution. [4]

(b) Let $X_t = A\cos(\pi t/3) + B\sin(\pi t/3) + Y_t$, where $Y_t = Z_t + 2.5Z_{t-1}$, $Z_t \sim \mathrm{WN}(0, \sigma^2)$. It is given that A, B are uncorrelated random variables with mean 0 and variance v^2 which are also uncorrelated with $\{Z_t\}$. Find acvf and the spectral distributions of $\{X_t\}$. [6]

Satyaki Mazumder