PH4105/5105 End-semester examination

Full marks : 50 **Time : 3 hrs**

Attempt any 5 questions

Q 1) a) Show that the integral curves of the vector field

$$X = xy\frac{\partial}{\partial x} + \frac{1 - x^2 + y^2}{2}\frac{\partial}{\partial y}$$

are circles centered on the X axis.

b) Prove that the Lie derivative $\pounds_X Y$ of a vector field Y with respect to X is given by $[X, Y]$.
$[6 + 4]$

Q 2) a) The connection coefficients are defined in local coordinates through

$$\nabla_{\frac{\partial}{\partial x^i}}\left(\frac{\partial}{\partial x^j}\right) \equiv \Gamma^k{}_{ji}\frac{\partial}{\partial x^k}$$

Find the transformation law connecting the connection coefficient functions under a change of coordinates from $(x^1, x^2, \ldots, x^n)$ to $(x'^1, x'^2, \ldots, x'^n)$.

b) Consider $M = \mathbb{R}^2$ with the standard sub-atlas comprising of the chart $(\mathbb{R}^2, \mathrm{id})$ with the trivial connection $\Gamma^k{}_{ij} = 0$. Find the connection coefficients if you use another chart with coordinates (r, θ) related to the original coordinates (x, y) by

$$x = r\cos\theta, \qquad y = r\sin\theta$$

$[5 + 5]$

Q 3) The exterior derivative $d\omega$ of a k-form ω is defined to be a $(k+1)$-form by

$$\begin{aligned}
&(k+1)\ d\omega\,(X_1, X_2, \ldots, X_{k+1}) \\
&\equiv \sum_{i=1}^{k+1}(-1)^{i+1}X_i\left(\omega\left(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1}\right)\right) \\
&\quad + \sum_{i<j}(-1)^{i+j}\omega\left([X_i, X_j], X_1, \ldots \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}\right)
\end{aligned}$$

Ananda Dasgupta

Where the "hat" denotes that the vector field wearing it is dropped from the list!

a) From this definition show that $d\omega$ is completely anti-symmetric.

b) Again, directly from this definition show that $d^2\alpha = 0$ where α is a 2-form. [5 + 5]

Q 4)a) Show that the torsion T defined by

$$T(X,Y) \equiv \nabla_X Y - \nabla_Y X - [X,Y], \qquad \forall X, Y \in \mathfrak{X}(M)$$

obeys

$$T(fX,Y) = fT(X,Y)$$

b) If the vector fields X and Y are given in local coordinates by $X = X^i \frac{\partial}{\partial x^i}$ and $Y = Y^i \frac{\partial}{\partial x^i}$, respectively, then show that

$$T(X,Y) = -2X^i Y^j \Gamma^k{}_{[ij]} \frac{\partial}{\partial x^k}$$

c) For a particular torsion-free connection on $\mathbb{R}^2$ with standard coordinates (x,y) we have

$$\nabla_X Y = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

where $X = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}$ and $Y = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$. Find $\nabla_Y X$. [3 + 3 + 4]

Q 5) a) The curvature tensor R, defined by

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]},$$

satisfies (with X, $Y$and Z being vector fields expressed in local coordinates)

$$R(X,Y)Z = X^i Y^j Z^k R^m{}_{kij} \left(\frac{\partial}{\partial x^m}\right)$$

Deduce the expression for $R^m{}_{kij}$ in terms of the connection functions $\Gamma^k{}_{ij}$.

b) Prove the Bianchi identity :

$$\begin{aligned}
&\nabla_X (R(Y,Z)W) + \nabla_Y (R(Z,X)W) + \nabla_Z (R(X,Y)W) \\
&= R(Y,Z)\nabla_X W + R(Z,X)\nabla_Y W + R(X,Y)\nabla_Z W \\
&\quad + R([Y,Z],X)W + R([Z,X],Y)W + R([X,Y],Z)W
\end{aligned}$$

[5 + 5]

Q 6) The connection 1-forms, torsion 2-forms and curvature 2-forms are defined by

$$\nabla_X e_i \equiv \Gamma^j{}_i (X)\, e_j, \qquad T^i (X,Y) \equiv \frac{1}{2}\theta^i (T(X,Y)), \qquad \mathcal{R}^i{}_j (X,Y) = \frac{1}{2}\theta^i (R(X,Y)\, e_j)$$

where $(\theta^1, \theta^2, \ldots, \theta^n)$ are dual to the (possibly non-holonomic) basis vector fields $(e_1, e_2, \ldots, e_n)$, $\theta^i (e_j) = \delta^i_j$.

a) Prove that

$$\nabla_X \theta^i = -\Gamma^i{}_j (X)\, \theta^j$$

b) Prove the two Cartan structure equations :

$$T^i = d\theta^i + \Gamma^i{}_j \wedge \theta^j$$
$$\mathcal{R}^i{}_j = d\Gamma^i{}_j + \Gamma^i{}_k \wedge \Gamma^k{}_j$$

[2 + 8]

Q 7) The metric tensor for a space $M = \{(x,y) \,:\, y > 1\}$ is given by

$$g = \frac{y-1}{y} dx \otimes dx - \frac{y}{y-1} dy \otimes dy$$

a) Find the non-zero Christoffel symbols for this metric.

b) Use the Cartan structure equations to find all the non-zero components of Riemann curvature tensor. [5 + 5]