Indian Institute of Science Education & Research

Final Examination - MA3102 Algebra I

Date : 6th December 2018

INSTRUCTIONS

This is a **closed-book** exam.

You have 3 hours.

- The examination is scored out of 50 **points**.
- The exam has **TWO** parts. Part I has **five** short problems. If you attempt all five, then you must indicate the **four** you want graded. Part II has **three** long problems.

Good luck!

06/12/18

Answers without justification will receive no credit.

PART I [25=5+5+5+5+5 points]

DO ANY FOUR PROBLEMS

Problem 1 Define a cyclic group. Show that G is a union of its proper subgroups if and only if G is not cyclic.

Problem 2 Let H and K be subgroups of a finite group G. Show that $|H \cap K| \cdot |HK| = |H| \cdot |K|$.

Problem 3 Let G be a group of size p^r, where p is a prime and $r \geqslant 1$. Show that $Z(G) \neq \{e\}$.

Problem 4 Prove or disprove: *There is an isomorphism between* $\mathbb{Q}$ *and* $\mathbb{Z} \times \mathbb{Z}$.

Problem 5 Show that D_6 and $S_3 \times \mathbb{Z}_2$ are isomorphic.

PART II [30= 10+ 10+ 10 points]

You may use a theorem proved in class by stating it carefully. Any other result you use has to be proven.

Problem A Let G be a group of size 3102.
(i) Show that there is a unique 47-Sylow subgroup.
(ii) Show that there is a unique 11-Sylow subgroup.
(iii) Show that there is a cyclic subgroup H of size 47×11 which is normal.
(iv) Show that there is a subgroup of index 2 in G.

Problem B Consider the group S_5.
(i) Count the number of elements of order 5 in S_5.
(ii) Show that any two 5-cycles are conjugate in S_5.
(iii) Count the number of conjugacy classes among 3-cycles in A_5.

Problem C This question concerns group actions.
(i) Show that an action $\varphi : \mathbb{Z}_k \times \{1, \ldots, n\} \to \{1, \ldots, n\}$ determines an element $\sigma \in S_n$ satisfying $\sigma^k = \text{id}$.
(ii) Suppose that $\sigma \in S_n$ is of order k. Construct an action $\varphi : \mathbb{Z}_k \times \{1, \ldots, n\} \to \{1, \ldots, n\}$ using σ.