Mid Semestral Examination

MA4103

Instructor: Dr. Soumya Bhattacharya

The maximum number of points which you may score is 20. Please justify your answers completely. No points will be given for answers lacking justification.

Problem 1. 1. Prove or disprove: If $R[X]$ is a PID, then R is necessarily a field.
2. Let R be a commutative ring with identity such that every element $r \in R$ satisfies

$$r^n = r$$

for some $n > 1$. Show that every prime ideal in R is maximal.
3. Let $R = \mathbb{Z}/4\mathbb{Z}$. Determine all the units in the ring $R[x]$.

(3 + 4 + 5)

Problem 2. Let R be an integral domain and let M be a module over R.
1. Let M_1, N_1, M_2 and N_2 be submodules of M with $M_1 \cap M_2 = \{0\}$, $N_1 \subseteq M_1$ and $N_2 \subseteq M_2$. Show that

$$(M_1 \oplus M_2)/(N_1 \oplus N_2) \simeq (M_1/N_1) \oplus (M_2/N_2).$$

2. Let M_1 and M_2 be two submodules of M such that $M_1 \cup M_2$ is also a submodule of M. Show that either $M_1 \subseteq M_2$ or $M_2 \subseteq M_1$.
3. Let M a free module of rank n. Let S be a minimal set of generators of M over R. Is it possible for S to have more than n elements?
4. Let R be a PID and let M be finitely generated over R. Let $I \subseteq \operatorname{rad} 0$ be such that

$$I(\operatorname{Tor} M) = \operatorname{Tor} M.$$

Show that M is a free module.
5. Every abelian group G is naturally a $\mathbb{Z}$-module. Can this action be extended to make G into a $\mathbb{Q}$-module?

(3 + 3 + 2 + 4 + 2)

Soumya Bhattacharya

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4 p.m. - 5 p.m. September 20, 2018