Department of Mathematics and Statistics
Indian Institute of Science Education and Research Kolkata
End-Autumn Semester Examination 2018
Subject: Analysis-I (MA2101)
Date and Time of Exam: December 4, 2018; 2:00PM
Max. Time: 3 Hours **Full Marks: 50**

****

Instructions: Answer all the questions. The number inside square bracket at the right of each question indicates the marks for that question. The notations have their usual meanings. All intermediate steps must be shown in order to get full credit.

****

1. Verify whether the following sequences $\{x_n\}$ are convergent. If yes, find the value of $\lim_{n\to\infty} x_n$.

   (a) $x_n = \frac{[cn]}{n}$ where $[x]$ is the highest integer contained in x, where $c \in \mathbb{R}$. [3]

   (b) $x_{n+1} = \sqrt[3]{x_n + 6}$, $x_1 = 1$. [5]

   (c) Find all the cluster points of the sequence $\left\{\sin\left(\frac{n\pi}{4}\right)\right\}$. [2]

2. (a) State and prove Cauchy's second theorem for finding limit of a sequence. Hence or otherwise, evaluate $\lim_{n\to\infty} x_n$, where $x_n^n = \binom{2n}{n}$. [6]

   (b) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = ?$ [4]

3. Answer the following questions.

   (a) Find $\lim_{x\to 1} \sqrt{x^4 - 4x^3 + 5x^2 - 2x}$ [1]

   (b) Find $\lim_{x\to 1} \frac{|x-1|}{x-1}$ [1]

   (c) If possible, give two functions f and g, and an $a \in \mathbb{R}$ such that neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists but $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists. [2]

   (d) Find the point of discontinuities, if any, of the function f defined by [2]

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 2 & \text{if } x \in \mathbb{Q}^c \cap [0,1] \end{cases}$$

(e) Give example of a function which is continuous exactly at two points. [1]

(f) Let $D \subseteq \mathbb{R}$. Suppose $F : D \to [0,1]$ is a function which is nondecreaing and (at least) right continuous at every $x \in D$. Prove that F cannot have uncountable number of discontinuities. [3]

4. Answer the following questions.

(a) Let $D \subset \mathbb{R}$. Suppose $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be uniformly continuous on D. Is $f+g$ uniformly continuous on D? What can you say about uniform continuity of the function $f.g$? [2+2=4]

(b) Let S and T be two sets. Define $ST = \{ab : a \in S, b \in T\}$. Verify whether $\inf(ST) = \inf(S) \times \inf(T)$. Can we say that $\sup(ST) = \sup(S) \times \sup(T)$? [2+2=4]

(c) Let $A = \left\{\frac{n+1}{n-1}, n = 2,3,4,\ldots\right\} \cup \{0\}$. Find the derived set of A. [2]

5. Answer each of the following with proper justification. [10]

(a) What is the set of boundary points of $\mathbb{Q}$?

(b) Mention a point which lies in the intersection of the set of exterior points and the set of boundary points of $\mathbb{Q}$.

(c) If we have two sets having same boundary points then are the two sets always same?

(d) Consider the function $f : \mathbb{Q} \to \mathbb{R}$ be defined as $f(x) = 1$ for all $x \in \mathbb{Q}$. Is f continuous at the point 2?

****

04-12-18