PH4207 :: Quantum Information Processing
Final Examination (May 8, 2019)

Marks: 45, Duration: 2 hours 30 mins

1. A single qubit is represented by the point (0.6,-0.48,0.64) on the Bloch Sphere.

   (a) Is this a pure state? If it is, find the ket vector representing it.

   (b) What is the density matrix for this state?

   (c) Find the point on the Bloch Sphere that is obtained by acting on this state by the **Z** gate.

   [6]

2. Prove that the singlet state $|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ is rotational invariant, that is, that it takes the same form

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle_{\mathbf{u}}|-\rangle_{\mathbf{u}} - |-\rangle_{\mathbf{u}}|+\rangle_{\mathbf{u}}\right)$$

   for any direction $\mathbf{u}$, the states $|+\rangle_{\mathbf{u}}$ & $|-\rangle_{\mathbf{u}}$ being eigenstates of $\sigma\cdot\mathbf{u}$ [5]

3. Consider the two qubit state

$$\rho = p\,|\psi\rangle\langle\psi| + \frac{1-p}{4}\mathbf{I}$$

   where $|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. Here $\boldsymbol{I}$ is the density operator on the 2-qubit Hilbert space.

   (a) Having written down ρ in the computational basis, find the concurrence of this matrix.

   (b) Find the range of values of p for which the state is entangled.

   (c) Discuss the positivity (whether a positive map or a completely positive map) of partial transposition operation.

   [6]

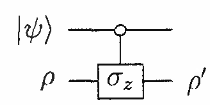
4. The GHZ state is given by,
   $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$.
   Suppose that Alice takes the first qubit and Bob takes qubits 2 and 3. Show that this scenario can be used for dense coding and give the dense coding protocol explicitly.
   [4]

5. (a) Using the Peres Horodecki partial transpose criterion, examine if the state, $|\psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)$, is entangled or not.

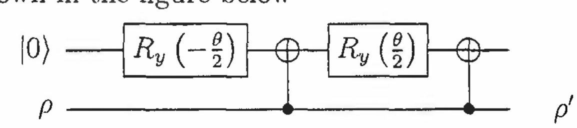
   (b) How can we say that the set of separable density matrices form a convex set. Using this property comment on the validity of Peres Horodecki partial transpose criterion to determine entanglement. [4]

6. Write down the Quantum Fourier Transform (QFT) operation on 3 qubits explicitly as a matrix in the computational basis and prove that it is a unitary operation. Compute explicitly the QFT of the 3 qubit state $\{|110\rangle\}$. Show that the state which results from the transform can be written as a product state of the three qubits. State why QFT is considered important in the area of code breaking (just state in one line, no detailed description required). [5]

7. A quantum circuit implementing the phase flip channel is



Now consider the Kraus operator $F_0 = \begin{bmatrix} 1 & 0 \\ 0 & cos\theta \end{bmatrix}$ & $F_1 = \begin{bmatrix} 0 & 0 \\ 0 & sin\theta \end{bmatrix}$ Now consider the quantum circuit shown in the figure below



Show that this circuit induces a quantum operation $\rho \rightarrow \rho' = F_0 \rho F_0^\dagger + F_1 \rho F_1^\dagger$ [6]

8. Consider a 2-level atom in a thermal radiation field. In such a situation the Lindblad-Master eqn. reads

$$\dot{\rho} = -\frac{i}{\hbar}[H,\rho] + \gamma(\bar{n}+1)(\sigma_+\rho\sigma_- - \frac{1}{2}\sigma_-\sigma_+\rho - \frac{1}{2}\rho\sigma_-\sigma_+) + \gamma\bar{n}(\sigma_-\rho\sigma_+ - \frac{1}{2}\sigma_+\sigma_-\rho - \frac{1}{2}\rho\sigma_+\sigma_-) \quad (1)$$

So that ω_0 is the frequency of the radiation that the atom will emit or absorb and $\sigma_\pm = \frac{1}{2}(\sigma_x + i\sigma_y)$ and $\bar{n}$ is the mean occupation number at temperature T $\left(\bar{n} = 1/\left(exp\left(\frac{\hbar\omega}{k_B T}\right) - 1\right)\right)$. Note that the Linblad operators (as done in the class) are $L_1 = \sqrt{\gamma(\bar{n}+1)}\sigma_+$ and $L_1 = \sqrt{\gamma\bar{n}}\sigma_-$ while L_1 drives transition $|1\rangle \rightarrow |0\rangle$, L_2 drives transition $|0\rangle \rightarrow |1\rangle$. Solve the Master equation (1). In particular discuss the approach to equilibrium. [9]