Duration 150 minutes Max Marks 50

1. Consider the Landau free energy with a cubic invariant

$$L = \frac{1}{2}\tilde{a}_2(T - T_c)\eta^2 + \frac{1}{3}a_3\eta^3 + \frac{1}{4}a_4\eta^4$$

   (a) Explain why it describes an abrupt (first-order) phase transition
   (b) Determine the transition temperature
   (c) Determine the temperature window where the free energy has two coexisting minima
   (d) Is there any constraint on the sign of a_4? Why?
   (e) Is there any constraint on the sign of a_3? Why?
   (f) Why can this free energy not describe the coarse-grained Ising model in absence of magnetic field?

(10 marks)

2. (a) The correlation length at the fixed point of the renrmalization group transformation must either be zero or infinity. Why?
   (b) Explain why and how the thermal eigenvalue of the RG transformation is related to the critical exponent ν?

(6 marks)

3. Let us assume that the singular part of the free energy close to the transition

$$f(t, h) = c_1 h^2 t^3 + c_2 t^2 h^3$$

   (a) Show that $f(t, h)$ is a generalized homogeneous function, i.e.,

$$f(\lambda^{a_t} t, \lambda^{a_h} h) = \lambda f(t, h) \qquad (1)$$

   and determine a_h and a_t. Here c_1, c_2 are constants and h and t denote the field and reduced temperature respectively.

   (b) Determine the value of the critical exponents β and γ, starting from the definition in Eq. (1).

(8 marks)

4. Derive the recursion relation for the coupling constant K for the one-dimensional Ising model with nearest neighbour interactions

$$\beta H = -K \sum_{\langle i,j \rangle} S_i S_j; \quad (h = 0)$$

undergoing decimation by a factor of 2.

(4 marks)

RBL

5. Renormalisation Group: Let the recursion relation for a thermal variable K under a scale change by a factor of 2 be of the form,

$$K' = K^3 + 3K^2(1 - K).$$

(a) Find the non-trivial ($K^* \neq 0, 1$) fixed point K^* for this recursion relation.

(b) Write down the linearized recursion relation. Is the fixed point relevant or irrelevant.

(c) What is value of the critical exponent ν?
[You can stop at the point where you will require a calculator to go further.]

(6 marks)

6. Gaussian fluctuations: Consider the Landau free energy

$$L = \int d^dx \left[\frac{1}{2}\gamma(\nabla\eta)^2 + a t\eta^2 + \frac{1}{2}b\eta^4\right]$$

Writing $\eta(r) = \bar{\eta} + \epsilon(r)$, where $\bar{\eta}$ is either one of the degenerate spontaneous mean field values of the order parameter below T_c, calculate the Landau free energy for the fluctuation $\epsilon(r)$. Work to quadratic order in ϵ and express your answer in the Fourier components of ϵ.

(6 marks)

7. Ginzburg criterion: Show that the requirement that $|\int d^dr\, G(r)| << \int d^dr\, \eta^2(r)$ arbitrarily close to the critical point leads to the value of the upper critical dimension

$$d_c = \frac{2\beta + \gamma}{\nu}.$$

What is the physical meaning of d_c?

(4 marks)

8. Lifshitz Transition: Assume that in the low temperature ordered phase (where $\eta \neq 0$), the free energy is given by the expression

$$L = \int \mathrm{d}^dr \;\{a(T - T^*)|\nabla\eta(r)|^2 + B|\nabla^2\eta(r)|^2\}.$$

Show that for $T < T^*$, the order parameter $\eta(r)$ is no longer spatially uniform. What is the wavelength of modulation of η? Explain how the system behaves as function of temperature.

(6 marks)