PH2201 (Physics IV) End-Semester Examination

Instructor: Siddhartha Lal Total: 50 marks Date: May 11, 2019

Write only your answer (any *one* of the choices A, B, C or D) for each multiple-choice question next to the "Answer:" below that question in the question paper. Marks will be deducted for incorrect answers to the multiple-choice questions. Please provide the detailed working out for questions 9-12 in your answer script. Hand in both the question paper and your answer script.

Name: Roll No.:

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1. Using the ground state wavefunction $\psi_1(x) = \sqrt{\frac{2}{a}}\sin(\frac{\pi x}{a})$ of a quantum particle within an infinitely deep 1D square-well potential of extent a, compute the expectation value of the operator $(\hat{x} + \frac{\hat{x}^3}{3})$? (2 marks)

   a. $a/2$
   b. a
   c. 0 (zero)
   d. $a/4$

   Answer:

2. Use the Schrödinger equation $H\psi = (-\frac{\hbar^2 d^2}{2m dx^2} + V(x))\psi$ together with the momentum operator $\hat{p}_x = -i\hbar\frac{d}{dx}$ to compute the commutator $[H, \hat{p}_x]$ for the potential $V(x) = \frac{a}{2}x^2 + \frac{b}{4}x^4$. (2 marks)

   a. $\frac{a}{2}x^2 + \frac{b}{4}x^4$
   b. $i\hbar(ax + bx^3)$
   c. $i\hbar(\frac{a}{2}x^2 + \frac{b}{4}x^4)$
   d. 0 (zero)

   Answer:

3. The state of a particle in an infinitely deep potential well and extent a is described by the wavefunction $\psi(x) = C\{[\sqrt{\frac{2}{a}}\cos(\frac{3\pi x}{a})] - \frac{1}{d}[\sqrt{\frac{2}{a}}\cos(\frac{5\pi x}{a})]\}$. What are the probabilities P_3 and P_5 that a measurement of the energy will yield E_3 and E_5? (2 marks)

   a. $P_3 = \frac{d^2}{a^2+d^2}$, $P_5 = \frac{a^2}{a^2+d^2}$
   b. $P_3 = \frac{9d^2}{9d^2+25}$, $P_5 = \frac{25}{9d^2+25}$
   c. $P_3 = \frac{25d^2}{25d^2+9}$; $P_5 = \frac{9}{25d^2+9}$
   d. $P_3 = \frac{d^2}{1+d^2}$, $P_5 = \frac{1}{1+d^2}$

   Answer:

4. For a quantum particle in a central potential, we have $\hat{\vec{L}}^2|l,m\rangle = l(l+1)\hbar^2|l,m\rangle$, $\hat{L}_z|l,m\rangle = m\hbar|l,m\rangle$. Using the relations $\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$ and $\hat{L}_\pm|l,m\rangle = \sqrt{l(l+1) - m(m\pm 1)}|l, m\pm 1\rangle$, compute the expectation values of (i) $\langle\hat{L}_x^2\rangle + \langle\hat{L}_y^2\rangle$ and (ii) $\langle\hat{L}_x^2\rangle - \langle\hat{L}_y^2\rangle$? (4 marks)

   a. (i) $\hbar^2[l(l+1) - m^2]$ (ii) 0 (zero)
   b. (i) 0 (zero) (ii) $\frac{\hbar^2}{2}[l(l+1) - m^2]$
   c. (i) $\frac{\hbar^2}{2}[l(l+1) - m^2]$ (ii) $\frac{\hbar^2}{2}[l(l+1) - m^2]$
   d. (i) $\frac{\hbar^2}{2}[l(l+1) - m^2]$ (ii) 0 (zero)

   Answer:

5. For an attractive delta function potential in one-dimension $V(x) = -\alpha\delta(x), \alpha > 0$ (such that the potential can support atleast one bound state), find the condition on the derivative on the wavefunction $d\psi/dx$ across the potential. Hint: integrate the Schrödinger equation from $-\epsilon$ to ϵ and take the limit of $\epsilon \to 0$. Recall that $\delta(x) = 0$ for $x \neq 0$, $\delta(x) = \infty$ for $x = 0$, such that $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$ and $\int_{-\infty}^{\infty} f(x)\, \delta(x-a) = f(a)$. (2 marks)
a. $(d\psi/dx)_{0+} - (d\psi/dx)_{0-} = 0$
b. $(d\psi/dx)_{0+} - (d\psi/dx)_{0-} = \hbar$
c. $(d\psi/dx)_{0+} - (d\psi/dx)_{0-} = -\frac{2m\alpha}{\hbar^2}\psi(x=0)$
d. $(d\psi/dx)_{0+} - (d\psi/dx)_{0-} = \alpha$

Answer:

6. The Schrödinger equation for the 1D harmonic oscillator is $(a_+a_- + \hbar\omega/2)\psi = E\psi$, where $a_\pm \equiv \frac{1}{2m}(-i\hbar\frac{d}{dx} \pm im\omega x)$ are the ladder operators. (i) Compute the result of $(a_+a_- + \hbar\omega/2)a_+\psi$. (ii) Use the answer you obtain in part (i) to obtain the energy eigenvalues E_n of this problem. (4 marks)
a. (i) $(E - \hbar\omega)a_+\psi$, (ii) $E_n = (n - 1/2)\hbar\omega$
b. (i) $(E + \hbar\omega)a_+\psi$, (ii) $E_n = (n + 1/2)\hbar\omega$
c. (i) $(E + \hbar\omega)a_+\psi$, (ii) $E_n = (n - 1/2)\hbar\omega$
d. (i) $(E - \hbar\omega)a_+\psi$, (ii) $E_n = (n + 1/2)\hbar\omega$

Answer:

7. For the n^{th} energy eigenstate ($n \geq 1$) of an infinitely deep one-dimensional square well potential of width a, the quantum revival time T_n is defined as the time over which the wave function of the particlein the n-th eigenstate returns to its original state: $\psi_n(x, T_n) = \psi_n(x, 0)$. What is the expression for $T_{n=3}$? (2 marks)
a. $T_3 = \frac{2ma^2}{3\pi\hbar}$
b. $T_3 = \frac{8ma^2}{9\hbar}$
c. $T_3 = \frac{4ma^2}{3\pi\hbar}$
d. $T_3 = \frac{4ma^2}{9\pi\hbar}$

Answer:

8. For a Hydrogen atom, the bound-state energy eigenvalues are given by $E_n = -\frac{13.6\text{eV}}{n^2}$. Upon being disturbed by a time-dependent perturbation, the excited states of this system acquire a finite lifetime $\Delta\tau_n$. (i) Compute an expression for $\Delta\tau_n$ (in units of eV) in the limit of $n >> 1$ by using the uncertainty relation $\Delta E_n \Delta\tau_n = \hbar$, where $\Delta E_n = E_n - E_1$, and $n > 1$. (ii) Compute $\Delta\tau_n$ in the limit of $n >> 1$ for a particle of mass m in a 1D infinitely deep well of extent a, with bound-state energy eigenvalues given by $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$. (4 marks)
a. (i) $\Delta\tau_n \sim \frac{\hbar n^2}{13.6(n^2+1)}$ (ii) $\Delta\tau_n \sim \frac{2ma^2}{\hbar\pi^2(n^2+1)}$
b. (i) $\Delta\tau_n \sim \frac{\hbar}{13.6(n^2-1)}$ (ii) $\Delta\tau_n \sim \frac{2ma^2(n^2+1)}{\hbar\pi^2}$
c. (i) $\Delta\tau_n \sim \frac{\hbar}{13.6}$ (ii) $\Delta\tau_n \sim \frac{2ma^2}{\hbar\pi^2 n^2}$
d. (i) $\Delta\tau_n \sim \frac{\hbar(n^2+1)}{13.6}$ (ii) $\Delta\tau_n \sim \frac{2ma^2(n^2+1)}{\hbar\pi^2}$

Answer:

9. A quantum particle of mass M confined on a circle of radius R has a Hamiltonian given by $H = \frac{p_\phi^2}{2I}$ where $p_\phi = -i\hbar\frac{d}{d\phi}$ is the angular momentum of the particle. $\phi \,\epsilon[0, 2\pi]$ (the angular coordinate defined on the circle) and p_ϕ are canonically conjugate to one another, $[\phi, p_\phi] = i\hbar$, and the moment of inertia of the particle is $I = MR^2$. (i) What are the (normalised) eigenfunctions and eigenvalues of H? (ii) What is the result of computing $U \, \frac{p_\phi}{\hbar} \, U^{-1}$, where the unitary transformation $U = e^{-ia\phi}$ (a is a constant)? (8 marks)

10. Using the raising and lowering (ladder) operators for the 1D quantum harmonic oscillator problem ($V(x) = \frac{1}{2}m\omega^2x^2$), $a_\pm = \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x)$, $[a_-, a_+] = 1$ together with the relations

$$a_+|n\rangle = \sqrt{n+1}|n+1\rangle \quad , \quad a_-|n\rangle = \sqrt{n}|n-1\rangle \quad , \quad \langle n_1|n_2\rangle = \delta_{n_1 n_2} \; ;$$

to compute (i) $\langle x^2 \rangle$ and (ii) $\langle p^2 \rangle$ for the state $|n\rangle$? (6 marks)

11. The three Pauli matrices are given by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; , \; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; , \; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; .$$

The spin operators for a spin-1/2 system are given by $\hat{S}_\alpha = \frac{\hbar}{2}\sigma_\alpha$, where $\alpha = (x, y, z)$. Further, $\hat{S}^2 = \frac{3\hbar^2}{4}\mathcal{I}$, $\mathcal{I}$ is the 2×2 identity matrix. (i) Compute the simultaneous eigenstates of $\hat{S}^2$ and $\hat{S}_z$, and thence the eigenvalues for these two operators. (ii) Now compute the simultaneous eigenstates of $\hat{S}^2$ and $\hat{S}_x$, and thence the eigenvalues for these two operators. (8 marks)

12. For the Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$ with a discrete set of eigenvalues (E_n) obtained from $\hat{H}|n\rangle = E_n|n\rangle$, use the quantity $\langle n| \left[\hat{H}, \hat{x}\right] |n'\rangle$ (where $|n'\rangle \neq |n\rangle$) to derive a relation between $\langle n|\hat{x}|n'\rangle$ and $\langle n|\hat{p}|n'\rangle$. How does this relation simplify for $|n'\rangle = |n\rangle$? (6 marks)