1. The interatomic Lennard Jones potential has a strongly repulsive part and an attractive part which is usually written as

$$V_{att}(r) \sim -\frac{c_0}{r^6}.$$

What is the reason for the dependence to be r^{-6}? Under what circumstance can this be r^{-3}?

(2 marks)

2. Consider the one dimensional Ising model with nearest neighbor coupling described by the Hamiltonian

$$H = -J \sum_{<i,j>} S_i S_j - h \sum_i S_i.$$

Here h is the applied external field. Given that the transfer matrix has the eigenvalues

$$\lambda_{1,2} = e^K \left[\cosh \beta h \pm \sqrt{\sinh^2 \beta h + e^{-4\beta J}} \right]$$

and the partition function $Z = \lambda_1^N + \lambda_2^N$, show that the system has a phase transition at $T = 0$ and at $T = 0$ only, i.e., the free energy is a non-analytic function of the magnetic field h at $T = 0$.

(4 marks)

3. Determine the critical exponent for the function $f(t) = At^2 e^{-t}$ as $t \to 0$.

(2 marks)

4. Consider the Ising model on a square lattice but now also including the next nearest neighbour ($< nnn >$) interactions.

$$H = -|J_1| \sum_{<nn>} S_i S_j - J_2 \sum_{<nnn>} S_i S_j - h \sum_{i=1}^{N} S_i$$

Here J_1 and J_2 are the two coupling constants. Solve the problem using a mean field approach. What should be the value of J_2 if the transition temperature has to become twice of what it is when there are no next nearest neighbour interactions?
(Note: Next nearest neighbours are along the diagonals.)

(7 marks)

5. Given that the number fluctuations in the grand canonical ensemble

$$\langle [N - \langle N \rangle]^2 \rangle = \frac{\langle N \rangle^2 k_B T}{V} \kappa_T, \tag{1}$$

where κ_T is the isothermal compressibility, show that

$$\int d^3 r \, G(r) = \frac{\langle N \rangle^2 k_B T}{V^2} \kappa_T. \tag{2}$$

Here $G(r)$ is the density-density correlation function. Write the relations analogous to equations (1) and (2) for spins.

(3 marks)

6. Give a heuristic argument for the existence of a phase transition at $T \neq 0$ for the 2D Ising model on a square lattice along with a rough estimate of the transition temperature.

(2 marks)