Indian Institute of Science Education and Research Kolkata
Department of Mathematics and Statistics
Geometry of Curves and Surfaces (MA3202)

Spring Sem
May 2, 2019

Time: 14:00–16:30 *End-Semester Exam* Marks: 50

1. A unit-speed curve $\gamma : (\alpha, \beta) \to \mathbb{R}^3$ with positive curvature is said to be a helix when all its tangent lines make a constant angle with a fixed line. Show that γ is a helix if and only if there exists $a \in \mathbb{R}$ such that $\tau(s) = a\kappa(s)$, for each $s \in (\alpha, \beta)$. [6]

2. Suppose that two surfaces S and $\tilde{S}$ are diffeomorphic and that S is orientable. Prove that $\tilde{S}$ is orientable. [6]

3. Let us consider the torus covered by the surface patch
$$\sigma(u, v) = ((a + r\cos u)\cos v, (a + r\cos u)\sin v, r\sin u),$$
where $0 < u < 2\pi$ and $0 < v < 2\pi$. Find all those points which are (i) Elliptic, (ii) hyperbolic, (iii) Parabolic and (iv) Planar. [6]

4. Let p be a point on a surface S. Show that the mean curvature at p is given by
$$H = \frac{1}{\pi}\int_0^{\pi} k_n(\theta)\,d\theta$$
where $k_n(\theta)$ is the normal curvature at p along a direction making an angle θ with a fixed direction. [6]

5. Let $U = \{(u, v) | v > 1\}$ and suppose $\sigma : U \to \mathbb{R}^3$ is a regular parametrized surface with $E = G = v^{-2}$ and $F = 0$. Verify the curve $\sigma \circ \gamma$ where
$$\gamma(s) = (a + r\tanh s, r\frac{1}{\cosh s}),$$
has unit speed and show that it is a geodesic. Here $a \in \mathbb{R}$ and $r > 0$ are constants ans s is assumed to belong in an interval for which $\gamma(s) \in U$. [6]

6. Let $\sigma : U \to \mathbb{R}^3$ be a patch of a surface S. Show that the image under the Gauss map of the part $\sigma(R)$ of S corresponding to a region $R \subset U$ has area
$$\int_R |K|\,dA_\sigma,$$
where K is a Gaussian curvature of S. [7]

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7. Let $h : S \to \mathbb{R}$ be a differentable function on a surface S, and let $p \in S$ be a critical point of h (i.e $dh_p = 0$). Let $w \in T_pS$ and let $\gamma : (-\epsilon, \epsilon) \to S$ be parametrized with $\gamma(0) = p$, $\gamma'(0) = w$. Set

$$H_p h(w) = \frac{d^2(h \circ \gamma)}{dt^2}\Big|_{t=0}.$$

a. Let $\sigma : U \to S$ be a parametrization of S at p and show that

$$H_p(u'\sigma_u + v'\sigma_v) = h_{uu}(p)u'^2 + 2h_{uv}(p)u'v' + h_{vv}(p)v'^2.$$

b. Let $h : S \in \mathbb{R}$ be the height function of S relative to T_pS,

$$h(q) = \langle q - p, N(p) \rangle, \quad q \in S.$$

Verify that p is a critical point of h and thus that the hessian is well-defined. Show that if $w \in T_pS$, $|w| = 1$, then $H_p(w) =$ normal curvature at p in the direction of w.[6+7=13]