MA 4201 : Fourier Analysis

Date : February 20, 2019 **Time : 15h30 - 17h00**

Problem 1.
Let (X, μ) be a σ-finite outer measure space, let $\phi \in C^1[0, \infty)$ be non-negative, increasing with $\phi(0) = 0$ and let $f : X \to \mathbb{R}$ be μ-measurable. Prove that

$$\int_X \phi(|f|)\, d\mu = \int_0^\infty \mu\{x \in X : |f(x)| > \lambda\} \phi'(\lambda)\, d\lambda.$$

[5 points]

Problem 2.
Let $f \in L^1(\mathbb{T})$ and let $g \in L^\infty(\mathbb{T})$. Prove that

$$\lim_{n \to \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) g(nt)\, dt = \hat{f}(0)\hat{g}(0).$$

[5 points]

Problem 3.
Let $f_n \in A(\mathbb{T})$ and let $f \in L^1(\mathbb{T})$ be such that

(i) $\|f_n\|_{A(\mathbb{T})} \leqslant 1$ for all $n \in \mathbb{N}$, and,

(ii) $(f_n)_{n \in \mathbb{N}}$ converges uniformly to f.

Prove that $f \in A(\mathbb{T})$ and $\|f\|_{L^1(\mathbb{T})} \leqslant 1$. [5 points]

Problem 4.
Let $f \in L^1(\mathbb{T})$ be defined as

$$f(t) := |t|, \text{ for all } t \in [-\pi, \pi].$$

Calculate Fourier coefficients of f. Find

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4}.$$

[5 points]