MA3203: Algebra II (Spring 2019)
End Semester, Duration: 150 minutes

Answer all the questions. The maximum possible points is 50.
No marks will be given without justification!

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1. True or false (justify your answers). $[3 \times 5]$

(a) There is an integral domain with 77 elements.

(b) The ideal $\mathcal{M} := (71X^3 + 21X^2 + 42X + 91) \subset \mathbb{Q}[X]$ is a prime ideal.

(c) There exists exactly one ring homomorphism (non-trivial) from $\mathbb{Z}$ to R, where R is a commutative ring with 1.

(d) Every degree-two field extension is separable.

(e) The splitting field of $g(X) = X^p - 1$ has degree p over $\mathbb{Q}$.

2. Answer the following questions. $[6 \times 5]$

(a) Prove that $(2+i) \subset \mathbb{Z}[i]$ is a maximal ideal. Also, determine the quotient field. [4+2]

(b) Show that $\frac{\mathbb{Z}[X]}{(X^2+5)}$ is not a PID. Let (R, d) be a Euclidean domain. Is $I := \{r \in R \mid d(r) > d(1)\} \cup \{0\}$ an ideal R? [3+3]

(c) Prove that a field extension $F \hookrightarrow K$ is algebraic iff every integral domain R with $F \hookrightarrow R \hookrightarrow K$ is a field.

(d) Find the degree of $\mathbb{Q}(\sqrt{3} + \sqrt{5})$ over $\mathbb{Q}(\sqrt{15})$.
Also, find a basis for $\mathbb{Q}(\sqrt{3} + \sqrt{5})$ over $\mathbb{Q}(\sqrt{15})$. [4+2]

(e) Prove that the rings $\frac{\mathbb{F}_3[X]}{(X^3+X^2+2)}$ and $\frac{\mathbb{F}_3[X]}{(X^3+2X+2)}$ are isomorphic, where $\mathbb{F}_3 = \frac{\mathbb{Z}}{3\mathbb{Z}}$.

3. (i) Define the Galois extension of fields. Also, define the Galois group. [2]
(ii) State the fundamental theorem of Galois theory. [2]
(iii) Verify the fundamental theorem of Galois theory for the splitting field of $X^4 + 1 \in \mathbb{Q}[X]$. [6]