Indian Institute of Science Education & Research Kolkata

Final Examination - MA4206
Algebraic Topology

2^{nd} May 201~~8~~9 10-12:30 p.m.

INSTRUCTIONS

This is a **closed-book** exam.

- The examination is scored out of 50 **points**.
- The exam has **THREE** questions and a **BONUS** question.
- Any score exceeding 50 will be counted as 50.
- Write clearly and with mathematical justification. Arguments lacking rigour will receive no credit.

Good luck!

2^{nd} May 2019

Problem 1 This question concerns covering spaces and covering maps.

(i) [3 **points**] Give a precise definition of the fundamental group $\pi_1(X, x_0)$ of a based space (X, x_0).

(i) [5 **points**] Consider the equivalence relation on S^3 that identifies (w, x, y, z) with $(-w, -x, -y, -z)$. Show that the quotient space is Hausdorff and path-connected.

(ii) [8 **points**] Show that if X is path-connected, then $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic. Give a counterexample when X is not path-connected.

(iii) [9 **points**] Let $p : \widetilde{X} \to X$ be a covering map such that $|p^{-1}(x)|$ is finite for any $x \in X$. Show that $\widetilde{X}$ is Hausdorff if and only if X is Hausdorff.

Problem 2 This question concerns lifting properties.

(i) [3 **points**] Define the homotopy lifting property for a map $\pi : \widetilde{X} \to X$.

(ii) [5 **points**] Show that if $\pi : (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering, then a nullhomotopic loop in X based at x_0 always lifts to a loop in $\widetilde{X}$.

Problem 3 This question concerns Seifert-van Kampen's Theorem and computations of fundamental groups.

(i) [3 **points**] State Seifert-van Kampen's Theorem with an explanation of the relevant term(s).

(i) [10 **points**] Let K be a compact subset of $\mathbb{R}^3$ and $x_0 \in \mathbb{R}^3 - K$. Show that the inclusion $\iota : \mathbb{R}^3 - K \hookrightarrow S^3 - K$ induces an isomorphism of fundamental groups $\iota_* : \pi_1(\mathbb{R}^3 - K, x_0) \to \pi_1(S^3 - K, x_0)$. [*You may use the fact higher dimensional spheres are simply connected.*]

(ii) [7 **points**] Consider the map $P : \mathbb{C} - \{0\} \to \mathbb{C} - \{1\}, P(z) = z^2 + 1$. Compute the map P_* at the level of fundamental groups. [*You are free to choose the basepoints of your liking. However, if you are using explicit generators, then state clearly what these loops are at the outset.*]

Bonus Problem [6 points] For a positive integer $n \geq 3$, let X_n be the quotient space obtained from the closed unit disk $\mathbb{D}^2$ by identifying $z \in \partial\mathbb{D}^2$ with $e^{\frac{2\pi i}{n}} z$.

(a) Compute the fundamental group of X_n.

(b) Show that X_n is not a surface.