PH3203 End-semester examination

Time : 2hr 30 mins | Full Marks :50

Group A

Answer any four questions $(4 \times 5 = 20)$

Q 1) Prove that for a 1-D system in an energy eigenstate

$$\langle T \rangle = \frac{1}{2} \left\langle x \frac{dV}{dx} \right\rangle$$

where $T = \frac{p^2}{2m}$ is the kinetic energy operator. If the energy eigenvalue of a 1-D quantum system with a potential $V \propto x^n$ is E, what is the expectation value of the potential? [4+1]

Q 2) Show that the energy correction up to (and including) the second order in perturbation theory of the ground state of the harmonic oscillator due to a perturbation αx^5 of a harmonic oscillator is negative. *You don't have to calculate the correction - just prove that it is negative.* [5]

Q 3) Let $U_0(t, t_0)$ be the time evolution operator for the Hamiltonian $\hat{H}_0 = \frac{1}{2}(\hat{p}^2 + \hat{q}^2)$ obeying the initial condition $U_0(t_0, t_0) = \mathcal{I}$. Calculate $U_0^\dagger(t, t_0)\,\hat{q}\,U_0(t, t_0)$ and $U_0^\dagger(t, t_0)\,\hat{p}\,U_0(t, t_0)$. [5]

Q 4) In cylindrical polar coordinates (ρ, θ, z) the laplacian operator is given by

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\right) + \frac{1}{\rho}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

Use the Ansatz $\psi(\rho, \theta, z) = R(\rho)\,\Theta(\theta)\,Z(z)$ to separate the Helmholtz equation $(\nabla^2 + k^2)\,\psi = 0$ into three ordinary differential equations. [5]

Q 5) Consider two harmonic oscillator systems with lowering and raising operator $a_+, a_+^\dagger$ and $a_-, a_-^\dagger$ respectively. Determine the commutation relations obeyed by the operators

$$A \equiv \frac{\hbar}{2}\left(a_+^\dagger a_+ - a_-^\dagger a_-\right), \ B \equiv \frac{\hbar}{2}\left(a_+^\dagger a_- + a_-^\dagger a_+\right), \ C \equiv \frac{\hbar}{2i}\left(a_+^\dagger a_- - a_-^\dagger a_+\right)$$

[5]

Q 6) The Berry phase acquired by an instantaneous Hamiltonian eigenstate upon a cyclic evolution of the parameters along a closed curve C in parameter space is given by

$$\Gamma = -i \oint_C \langle \phi_n (\boldsymbol{R}) | \nabla_{\boldsymbol{R}} | \phi_n (\boldsymbol{R}) \rangle \cdot d\boldsymbol{R}$$

Determine the change in this phase when the energy eigenstates are redefined by

$$|\phi_n (\boldsymbol{R})\rangle \rightarrow |\phi_n (\boldsymbol{R})\rangle' = e^{-i\mu_n(\boldsymbol{R})} |\phi_n (\boldsymbol{R})\rangle$$

Comment on the significance of this result. [5]

Group B

Anwer any three questions $(3 \times 10 = 30)$

Q 7) Prove the Feynman-Hellman theorem $\frac{\partial E_n}{\partial \lambda} = \langle \psi_n | \frac{\partial H}{\partial \lambda} | \psi_n \rangle$ where λ is any parameter that occurs in the Hamiltonian H and $|\psi_n\rangle$ is the normalized eigenket of H corresponding to the eigenvalue E_n.

Use it to determine the expectation values of $r^{-1}$and r^{-2} in the energy eigenstate ψ_{nlm} of the "radial" hydrogen Hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dr^2} - \frac{e^2}{4\pi\epsilon_0 r} + \frac{l\,(l+1)\,\hbar^2}{2mr^2}$$

The energy eigenvalue is

$$E_n = -\frac{m^2}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{1}{n^2}$$

[4+6]

Q 8) Find the constants α, β and γ such that the state

$$|\psi\rangle = \alpha\,|\phi_1\rangle + \beta\,|\phi_2\rangle + \gamma\,|\phi_3\rangle$$

is a simultaneous eigenstate of the operators J^2, J_z,J_1^2 and J_2^2 corresponding to the eigenvalues $3\,(3+1)\,,2,2\,(2+1)$ and $2\,(2+1)$ respectively. Here $|\phi_i\rangle$s are simultaneous eigenstates of J_1^2, J_2^2,$J_{1z}$and J_{2z} respectively. The respective J_{1z} and J_{2z} eigenvalues are $(2,0)$, $(1,1)$ and $(0,2)$ for the three states $|\phi_1\rangle$, $|\phi_2\rangle$ and $|\phi_3\rangle$, respectively.

Determine the expectation value of $\vec{J_1}\cdot\vec{J_2}$ in the state $|\psi\rangle$ [8+2]

Q 9) The first excited state of the harmonic oscillator Hamiltonian (in natural units)

$$H_0 = -\frac{d^2}{dx^2} + x^2$$

is given by $\psi = Axe^{-x^2/2}$. Find the energy (up to first order in perturbation theory) of the first excited state of the perturbed Hamiltonian

$$H = H_0 + \alpha x^6$$

Use the Dalgarno-Stewart approach to determine the first order corrected wave function [4+6]

Q 10) Consider two operators A and B obeying the commutation relations $[A, B] = B$. Consider the operators $U = e^{A+B}$ and $V = e^{\alpha A}e^{\beta B}$ where $\alpha, \beta \in \mathbb{C}$. Using the Baker-Hausdorff lemma calculate UAU^{-1},VAV^{-1}, UBU^{-1}, and VAV^{-1}. Hence determine the values of α and β for which $U = V$.[8+2]

Q 11) For the hard sphere potential

$$V = \begin{cases} \infty & \text{for } r \leq a \\ 0 & \text{for } r > a \end{cases}$$

the wave function is given by

$$\psi(r, \theta) = \sum_{l=0}^{\infty} i^i (2l+1) \left[j_l(kr) + ika_l h_l^{(1)}(kr) \right] P_l(\cos\theta) \qquad \text{for } r \geq a$$

with the boundary condition $\psi(a, \theta) = 0$. Find the phase shifts δ_l .

Use the phase shift δ_0 to determine the low energy differential and total scattering cross sections. [4+(4+2)]

Q 12) According to the first Born approximation, the scattering function for a localized potential V is given by the Fourier transform

$$f(\theta, \phi) = -\frac{m}{2\pi\hbar^2} \int e^{i(\vec{k}' - \vec{k}) \cdot \vec{r}'} V(\vec{r}')\, d^3 r'$$

Show that if the potential is spherically symmetric this reduces to

$$f(\theta) = -\frac{2m}{\hbar^2 q} \int_0^{\infty} rV(r) \sin(qr)\, dr$$

where $q = \left| \vec{k}' - \vec{k} \right|$.

Use this to calculate the differential and total scattering cross section for the Yukawa potential

$$V(r) = A \frac{e^{-\mu r}}{r}$$

[4+(4+2)]

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