IISER Kolkata **Final Semester Examination**

MA 4201 : Fourier Analysis

Date : May 4, 2019 **Time : 10h00 - 12h30**

Problem 1.
Show that

$$\left|\frac{1}{\sin t} - \frac{1}{t}\right| \leqslant \frac{\pi}{4}, \text{ for all } t \in \left(0, \frac{\pi}{2}\right).$$

[5 points]

Problem 2.
Calculate

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2+1}.$$

[8 points]

Problem 3. (Wallis Identity)
Prove that

$$\sin(\pi\lambda) = \pi\lambda \prod_{n\in\mathbb{N}} \left(1 - \frac{\lambda^2}{n^2}\right), \text{ for all } \lambda \in (-1, 1).$$

In particular, deduce that

$$\frac{2}{\pi} = \prod_{n\in\mathbb{N}} \left(1 - \frac{1}{4n^2}\right)$$

Hint : For $\lambda \in (-1, 1)$ with $\lambda \neq 0$, consider the function $f \in L^1(\mathbb{T})$ defined as

$$f(x) := \cos(\lambda x), \text{ for all } x \in [-\pi, \pi].$$

[10 points]

Problem 4.
Let $n \in \mathbb{N}$ and let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Prove that,

$$\lim_{k\to\infty} E_k(f)(x) = f(x), \text{ for all } x \in \mathbb{R}^n \text{ a.e.}$$

[10 points]

Problem 5.
Let $n \in \mathbb{N}$, and, for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, let $\mathcal{M}(f)$ denote the Hardy-Littlewood Maximal Function of f.

(i) Let us define

$$\mathcal{S} := \{(p,q) \in [1,\infty] \times [1,\infty] : \mathcal{M} \text{ is strong } (p,q)\}.$$

Find $\mathcal{S}$. [10 points]

(ii) Calculate $\mathcal{M}\left(\chi_{(0,1)}\right)$. [5 points]

Problem 6.
Let $1 \leqslant p, q, r \leqslant \infty$ satisfy

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

Prove that $f * g \in L^r(\mathbb{T})$, for all $f \in L^p(\mathbb{T})$ and $g \in L^q(\mathbb{T})$. [10 points]