INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH KOLKATA

Department of Mathematics and Statistics

End-semestral Examination, Spring 2019

ANALYSIS IV (MA 3204)

Date: May 07, 2019 Maximum Marks: 50 Time: 1400 – 1630

Note: You need to write precise statement of the theorem that you are using, provided that you are not asked to prove the theorem itself.

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(1) Let X be a set and $\mathcal{M} \subseteq \mathcal{P}(X)$ such that $\mathcal{M}$ is non-empty. Prove that if $\mathcal{M}$ is closed under complements and countable union of disjoint sets, the $\mathcal{M}$ is a σ-algebra. [3]

(2) Let V denotes a Vitali set in $\mathbb{R}$. Show that $V \times \{0\}$ is Lebesgue measurable in $\mathbb{R}^2$ but not Borel measurable. [6]

(3) Let X be a set and δ_{x_0} be the Dirac delta measure concentrated at $x_0 \in X$. Characterize the real valued functions f on X which are integrable over X with respect to δ_{x_0}. [3]

(4) Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose f_n be a sequence of non-negative measurable functions converging to f almost everywhere on X and $\int f \, d\mu = \lim \int f_n \, d\mu < \infty$.

(a) Show that $\int_E f \, d\mu = \lim \int_E f_n \, d\mu$ for all $E \in \mathcal{M}$.

(b) Show that (a) is not true if $\int f \, d\mu = \lim \int f_n \, d\mu = \infty$. [5 + 4 = 9]

(5) Let $(X, \mathcal{M}, \mu)$ be a measure space and f be a non-negative measurable function on X. Let

$$\lambda(E) = \int_E f \, d\mu$$

for $E \in \mathcal{M}$. Show that

(a) λ is a measure on $\mathcal{M}$ and

(b) for any non-negative measurable function g on X

$$\int g \, d\lambda = \int fg \, d\mu.$$

[4 + 4 =8]

(6) Let $(X, \mathcal{M}, \mu)$ be a measure space and $f \in L^1(X, \mu)$. Show that $\int_E f \, d\mu = 0$ for every $E \in \mathcal{M}$ if and only if $f = 0$ almost everywhere on X. [5]

(7) (a) State Dominated Convergence Theorem.

(b) Compute the following limit and justify your calculation:

$$\lim_{n\to\infty} \int_0^\infty (1 + \frac{x}{n})^{-n} x^{-\frac{1}{n}} \; dm.$$

[2 + 6 = 8]

(8) Prove or disprove: Every Cauchy sequence in $L^1(\mathbb{R}^d)$ converges almost everywhere to its limit function in $L^1(\mathbb{R}^d)$. [5]

(9) (a) State Fubini-Tonelli Theorem.

(b) Prove that for $a > 0$,

$$\int_{\mathbb{R}^d} \exp(-a\|x\|^2) \; dm = \left(\frac{\pi}{a}\right)^{d/2}.$$

Justify your calculations. [3 + 7 = 10]

(10) Let $f \in L^1(0,1)$ and let $g : (0,1) \to \mathbb{R}$ be defined as

$$g(x) := \int_x^1 \frac{f(t)}{t} \, dt, \text{ for all } x \in (0,1).$$

Prove that $g \in L^1(0,1)$, and

$$\int_0^1 g(t) \, dt = \int_0^1 f(t) \, dt.$$

[5]