PH 4207, Mid Semester Exam (Total Marks 25)
(22/02/19)

1. (a) Consider the unit vector on the Bloch sphere, given by $\hat{n} = (sin\theta\ cos\phi, sin\theta\ sin\phi, cos\theta)$.
Show explicitly using unitary rotation on the Bloch sphere, how can one go from the state $|0\rangle$ in the Bloch sphere to $\hat{n}$ in exactly one step. [4]

   (b) Suppose $R_y(\theta)$ and $R_z(\theta)$ are the rotation operators for a qubit about the y and z axes and suppose you are given a one qubit unitary operation W = $R_z(\alpha)R_y(\theta)R_z(\alpha)$. Show that W can be written as W = $A\sigma_x B\sigma_x$, where A and B are unitary operations given by A = $R_z(\alpha)R_y(\frac{\theta}{2})$ and B = $R_y(\frac{-\theta}{2})R_z(-\alpha)$. What does the operator AB correspond to? [3]

2. (a) Consider a quantum state of an electron consisting of a magnetic moment, $\vec{\mu} = \gamma\ \hbar\ \vec{\sigma}$, where, $\vec{\sigma} = \hat{i}\ \sigma_x + \hat{j}\ \sigma_y + \hat{k}\ \sigma_z$, evolving under a magnetic field. The Hamiltonian, H under a static magnetic field B applied along the x-axis is given by,

$$H = -\frac{1}{2}\hbar\ \omega\ \sigma_x,$$

   where, $\omega = \gamma\ B$ and γ is the gyromagnetic ratio. If initially the electron is in the state $|0\rangle = |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, find out the minimum time necessary to flip the spin from $|\uparrow\rangle$ to $|\downarrow\rangle$. Find also the time necessary to take the initial state, $|\uparrow\rangle$ to an arbitrary state, $\phi = \frac{|\uparrow\rangle + i|\downarrow\rangle}{\sqrt{2}}$. Why this state cannot be distinguished from the state $|\uparrow\rangle$. [4]

   (b) Calculate the expectation value of $\vec{\mu}$ in an arbitrary state after a time t since the evolution has started from it's initial state. Please give a physical interpretation of the result. [3]

3. Consider the singlet state,

$|\psi\rangle = \frac{1}{\sqrt{2}}[|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2]$, where $|\uparrow\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\downarrow\rangle = |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are two orthogonal states.

(a) Show that the quantum mechanical expectation value, $E(n,m) = \langle\psi|(\vec{\sigma}^A.\hat{n})(\vec{\sigma}^B.\hat{m})|\psi\rangle$, is given by $E(n,m) = -\hat{n}.\hat{m}$, where, $\hat{m}$ and $\hat{n}$ are unit vectors and the superscript A and B refer to Hilbert spaces of Alice and Bob (as done in class). [3]

(b) The CHSH-inequality is given by, $|E(n,m) - E(n,m')| + |E(n',m') + E(n',m)| \leq 2$. Find the angles where the inequality is maximally violated. Interpret the result. [2]

4. Consider the 4 particle state,

$$|W\rangle = \frac{1}{2\sqrt{2}}(|1100\rangle + \sqrt{2}|1010\rangle - |1001\rangle + |0011\rangle - \sqrt{2}|0101\rangle + |0110\rangle).$$

Perform the the Schmidt decomposition of $|W\rangle$, such that you partition the system into two sub-systems (13|24), i.e., $|W\rangle = \sum_i \lambda_i |e_i\rangle_{13}|e_i\rangle_{24}$. Subsequently, evaluate the entanglement between these two sub-systems (13|24). How many e-bits of entanglement do you obtain? Is it a maximally entangled state? [6]