Copy of Final Exam
MA 4203 (Differential Geometry)
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(Instructor)

Final Exam

Differential Geometry

2 – 4:30 pm, May 5, 2019

A) Full justification is required for all the problems.

B) In problems (2) and (4), you may use the fact that integration of a top form on a manifold does not change if we remove a subset of measure zero from the manifold (i.e., a point from a circle, or a circle from a sphere etc.).

1. a) (2pts) Let $\sigma : M \to N$ be a smooth map between smooth manifolds. If ω is a differential form on N, then define $\sigma^*(\omega)$.

   b) (5 pts) Let f be a smooth function on N, so that df is a 1-form on N. Prove that
   $$\sigma^*(df) = d(\sigma^*(f))$$

   c) (5 pts) Show that σ^* commutes with exterior differentiation, i.e.,
   $$\sigma^*(d\omega) = d(\sigma^*(\omega))$$
   where ω is any differential form on N.

2. Let $\mathbf{S}^1$ be the unit circle in $\mathbf{R}^2$. Let $\omega = xdy - ydx$. It is a 1-form on $\mathbf{R}^2$. Its restriction to $\mathbf{S}^1$ defines a 1-form on $\mathbf{S}^1$, which we continue to denote by ω.

   a) (5 + 2 pts) Let $p \in \mathbf{S}^1$ and let v_p be the tangent vector to $\mathbf{S}^1$ at p, pointing in the counterclockwise direction and having length 1. Show that $\omega(v_p) = 1$ for all p. Conclude that ω is invariant under rotations (i.e., if σ is any rotation, then $\sigma^*(\omega) = \omega$).

   b) (5 pts) Let $f : \mathbf{R} \to \mathbf{S}^1$ be the map $\theta \mapsto e^{i\theta}$. Compute $f^*(\omega)$.

   c) (3 pts) Compute $\int_{\mathbf{S}^1} \omega$ (here $\mathbf{S}^1$ is provided with counterclockwise orientation, i.e., suitable restrictions of f form oriented charts for $\mathbf{S}^1$).

3. We retain the notation of the previous problem.

   a) (2 pts) Let η be any 1-form on $\mathbf{S}^1$. Show that there exists a unique (smooth) function f on S^1 such that $\eta = f\omega$.

   b) (6 pts) View $\mathbf{S}^1$ as a subset of $\mathbf{C}$. Let $\sigma : \mathbf{S}^1 \to \mathbf{S}^1$ be the map $z \mapsto z^2$ and let $\eta = \sigma^*(\omega)$. Compute the function f in (a).

4. Let $\mathbf{S}^2$ be the unit sphere in $\mathbf{R}^3$ and let
   $$\omega = xdy \wedge dz - ydx \wedge dz + zdx \wedge dy$$
   We view ω as a 2-form on $\mathbf{S}^2$.

Let H be the (open) upper hemisphere in $\mathbf{S}^2$ and let D be the open unit disk in the xy plane. Note that H projects differmorphically to D (projection taken along z direction). Let $\sigma : D \to H$ be the inverse of the projection map.

a) (4 pts) At each $p \in D$, let p_x, p_y denote that unit vectors at p in the x and y directions (i.e., $p_x = \partial/\partial x$ at p etc.). Compute the images of p_x, p_y under σ. Note that they are tangent vectors to H at $\sigma(p)$.

b) (5 pts) Using (a), or otherwise, compute $\sigma^*(\omega)$.

c) (6 pts) Using (b), or otherwise, compute $\int_{\mathbf{S}^2} \omega$ (you are free to choose your orientation).