MA 5215: FINAL EXAM

IISER Kolkata, Spring Semester, 2018-19.
Instructor: **Imran H. Biswas**
Date: **03.05.2019** Time: 10:00-12:30 Full marks: 50

(Q.1) 10

A Markov chain $\{X_n, n \geq 0\}$ with states $0, 1, 2$ has the transition probability matrix

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

If $P\{X_0 = 0\} = P\{X_0 = 1\} = \frac{1}{4}$, find $E[X_3]$.

(Q.2) 8+7

a.) A transition probability matrix $\mathbf{P}$ is said to be doubly stochastic if the sum over each column equals one; i.e.

$$\sum_i P_{ij} = 1, \text{ for all } j.$$

If a Markov chain with such a transition probability matrix is irreducible and aperiodic and consists of $M+1$ states $0, 1, 2, 3, \ldots.., M$, show that $\lim_{n\to\infty} P_{ij}^{(n)} = \frac{1}{M+1}$.

b.) Let $\{Y_n\}$ be the sum of n independent rolls of a fair die. Find

$$\lim_{n\to\infty} P(Y_n \text{ is a multiple of } 13).$$

(Q.3) 10

Suppose that $\xi_1, \xi_2, \ldots..$ are mutually independent family of random variables such that $E(\xi_i) = 0$ and $E(\xi_i^2) = \sigma_i^2$. Let $S_n = \sum_{i=1}^n \xi_i$ and $s_n^2 = \sum_{i=1}^n \sigma_i^2$. Show that $\{S_n^2 - s_n^2\}$ is a martingale with respect to the filtration generated by $\{\xi_n\}$.

(Q.4) 8+7

a.) Let $X_1, X_2, X_3, \ldots.$ be an independent sequence of random variables and Y is another random variable that is $\sigma(X_n, X_{n+1}, \ldots.)$-measurable for all n. Then show that Y is almost surely a constant.

b.) Let X be a square integrable random variable. Show that $\mathrm{Var}(X) = \min_{c\in\mathbb{R}} E(|X - c|^2)$. Moreover, if $P(a \leq X \leq b) = 1$ then show that $\mathrm{Var}(X) \leq \frac{(b-a)^2}{4}$.

Arima Chakraborty 3/5/19.

On behalf of Dr. Imran Habib Biswas

Good luck !!!