ENDSEM EXAM: PH4208, SOFT CONDENSED MATTER PHYSICS, SPRING 2019

MARKS: 40, TIME: 2.5 HOURS, 10 MAY 2019. 10:00 AM

ANSWER A MINIMUM OF 40 MARKS FROM THE QUESTIONS. THE WEIGHT OF THIS EXAM IS 40/100 (10/100 OF ENDSEM HAS BEEN TAKEN IN THE LAST CLASS TEST). IF RUNNING SHORT IN TIME, ANSWER AS MUCH AS YOU CAN. WRITE EVERY STEP OF YOUR CALCULATION. GOOD LUCK!

1. Consider N Ising spins in a 2-dimensional lattice. Find the equation of state for the magnetization m in the mean-field using the **microcanonical** ensemble. Draw the solutions for both $T > T_c$ and $T < T_c$. Take the exchange interaction parameter to be J.

[**4 marks**]

2. Consider a 2-dimensional lattice made of similar polygons of arms of equal length. Which types of polygons are allowed if they are to fill the space? Give reasons to your answer.

[**2 marks**]

3. The partition function for a van der Waal's gas is:

$$Z_N = \frac{1}{N!}\left(\frac{2\pi m k_B T}{h^2}\right)^{3N/2}[(V - bN)e^{aN/Vk_BT}]^N$$

. Find the critical temperature, density, and pressure in terms of a and b. What are these values for an ideal gas?

[**4 marks**]

4. Show that, in the thermodynamic limit, $S(\mathbf{q}) = \langle n\rangle[1 + \int \langle n\rangle g(\mathbf{x})e^{-i\mathbf{q}\cdot\mathbf{x}}d^dx]$ for homogeneous and isotropic fluids. Further, show that for ideal gas $S_{nn}(\mathbf{q}) = \langle n\rangle$. Symbols have usual meaning.

[**4 marks**]

5. Identify the dispersion relation for photons and electrons from the following:

$$\epsilon = \hbar\omega, \epsilon = \frac{\hbar^2 k^2}{2m_e}. \tag{0.1}$$

a) For visible light, $\epsilon \sim 1$eV. What are the length scales that can be probed by scattering of visible light?

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b) Typically in soft matters, what is the length-scale one tends to probe? What will be the temperature corresponding to a beam of neutrons of wavelength 1 angstrom?

[4 marks]

6. In the lattice model of a binary mixture (solvent and solute have equal specific volume), show that the mean-field internal energy can be written as $E = (1/2)Nz\Delta\epsilon\phi^2$, where N, z, ϕ are, respectively, number of lattice points, lattice co-ordination number, and solute volume fraction, and $\Delta\epsilon = \epsilon_{pp} + \epsilon_{ss} - 2\epsilon_{ps}$, where ϵ_{pp}, ϵ_{ss}, ϵ_{ps} are interactions, respectively between, solute-solute, solvent-solvent, and solute-solvent pairs.

Find one-to-one correspondence of the mean-field free energy between the Ising spin system and a binary mixture of small molecules. Identify equivalent applications for the mean-field concept in both systems as well.

Find the equation of the coexistence curve for the above binary mixture in terms of χ and ϕ. [Hint: Use the symmetrical form of the free energy density as function of ϕ.]

[12 (4+4+4) marks]

7. Show that the radial distribution function, $g_F(r)$, for the random, isotropic polymer, goes as $r^{-(d-2)}$.

How will $S(q)$ depend on q?

[2 marks]

8. Explain, in your own words and sketches, the origin of the formation of planar bilayer lamellae made of amphiphilic molecules.

[3 marks]

9. If $f(\phi, T)$ is the free energy density of a binary solution, find the expression for the chemical potential of the solute molecules as a function of the free energy density.

Further, show that $f(\phi, T)$ must have a concave upward part if we have to have a phase separation in the system?

[6 (3+3) marks]

10. Consider the mixture of polymer and solvent in the form of a gel (many single chains crosslinked to form one giant molecule floating in the solvent as a mesh). The Flory-Huggins free energy density for a gel is given by

$$f(\chi, S, \phi_0, \phi)/k_BT = (1-\phi)\log(1-\phi) + \chi\phi(1-\phi) + \frac{3}{2}S\phi_0^3\left[\left(\frac{\phi}{\phi_0}\right)^{1/3} - \left(\frac{\phi}{\phi_0}\right)\left(1 - \frac{1}{3}\log\left(\frac{\phi}{\phi_0}\right)\right)\right]. \quad (0.2)$$

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where $\chi = \Theta/2T$, where Θ is a constant, and S, ϕ_0 are parameters in the theory.

a) Expand f in powers of ϕ (note: $\phi < 1$) to up to order 3. Then show that the expression for osmotic pressure would be:

$$\Pi(\phi << 1) = k_B T\left[(\frac{1}{2} - \chi)\phi^2 + \frac{1}{3}\phi^3 - S\phi_0^3\left[\left(\frac{\phi}{\phi_0}\right)^{1/3} - \frac{\phi}{2\phi_0}\right]\right]. \qquad (0.3)$$

[Hint: keep the S term unchanged in the expansion over ϕ. Use, $\log(1-x) = -x - x^2/2 - x^3/3 - ..$]

b) What is the criterion for the gel to be in equilibrium with the solvent? Show that the minimization of the total free energy F leads to the condition that $\Pi = 0$.

[6 (3+3) marks]

Formulae you may find useful (but may not require to use all of them):

1. The first law of thermodynamics:

$$dE = TdS - PdV + \mu dN$$

2.

$$C_v = \left(\frac{\partial E}{\partial T}\right)_{N,V}, C_p = \left(\frac{\partial(E+PV)}{\partial T}\right)_{N,P}$$

3.

$$F = E - TS, \qquad G = F + PV,$$

where F, G are Helmholtz and Gibbs free energies respectively.

4.

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1$$

5.

$$\Gamma(\nu) = \int_0^\infty e^{-x}x^{\nu-1}dx; \qquad \nu > 0$$

$$\Gamma(n) = (n-1)!, \qquad \Gamma(n+1/2) = (n-1/2)(n-3/2)...3/2.1/2.\sqrt{\pi}, \qquad \Gamma(1/2) = \sqrt{\pi}$$

6. Hyper-volume and hyper-surface area of n dimensional sphere of radius R:

$$V_n(R) = \frac{\pi^{n/2}}{(n/2)!}R^n, \qquad S_n(R) = \frac{2\pi^{n/2}}{\Gamma(n/2)}R^{n-1}$$

8. Stirling's formula:

$$\ln(N!) \simeq N\ln N - N, \qquad N \gg 1$$

9. 1 eV $= 1.6 \times 10^{-19}$ J, $h = 6.62 \times 10^{-34}$m^2.kg/s, $c = 3 \times 10^8$m/s, $m_e = 9.11 \times 10^{-31}$kg, $m_p = 1.67 \times 10^{-27}$kg, $k_B = 1.38 \times 10^{-23}$ m^2 kg s^{-2} K^{-1}.

17. Mean thermal wavelength, $\lambda = h/(2\pi m k_B T)^{1/2}$.

18. The grand potential in GCE is given by: $d\tilde{A} = -SdT - Nd\mu - PdV$.

19. $\ln(1+x) = x - x^2/2 + x^3/3 - x^4/4 + ..$

21. Single particle (non-relativistic classical gas) energy levels:

1d: $\epsilon = (h^2/8mL^2)n_x^2$

2d: $\epsilon = (h^2/8mL^2)(n_x^2 + n_y^2)$

3d: $\epsilon = (h^2/8mL^2)(n_x^2 + n_y^2 + n_z^2)$

23. $\int_0^\infty e^{-\alpha y^2} y^\nu dy = \frac{1}{2\alpha^{(\nu+1)/2}}\Gamma\left(\frac{\nu+1}{2}\right), \qquad \nu > -1$

Arindam Kundagrami
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