PH3203 Mid Semester Examination

Total marks: 20 **Time: 90 minutes**

Answer any two questions

Q 1a) Define the operator valued function

$$F(\lambda) = e^{\lambda A} B e^{-\lambda A}$$

where A and B are operators, which do not, in general, commute.

(i) Prove that

$$\frac{dF}{d\lambda} = e^{\lambda A} [A,\, B]\, e^{-\lambda A}$$

(ii) Hence shown that

$$e^{A} B e^{-A} = \sum_{n=0}^{\infty} \frac{[A, B]_n}{n!}$$

where $[A, B]_0 \equiv B$ and $[A, B]_{n+1} \equiv [A, [A, B]_n]$.

b) Let $V = e^{aJ_+} e^{bJ_-} e^{cJ_3}$ where a, b and c are numbers and the operators $J_3, J_\pm$ are the usual angular momentum generators obeying the commutation relations

$$[J_3, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = 2J_3$$

(i) Show that

$$VJ_+V^{-1} = e^c \left\{ -2b\,(1+ab)\,J_3 + (1+ab)^2\,J_+ - b^2 J_- \right\}$$

(ii) Determine similar closed form expressions for VJ_-V^{-1} and VJ_3V^{-1}. **[(1 + 3) + (2 + 4)]**

Q 2) a) Calculate the matrix element

$$\langle j_1, m_1; j_2, m_2 |\, \vec{J}_1 \cdot \vec{J}_2\, | j_1'.m_1'; j_2', m_2' \rangle$$

b) Determine all the Clebsch-Gordon coefficients for the addition of angular momenta corresponding to $j_1 = 1,\ j_2 = 1$. **[4 + 6]**

Q 3) a) State and prove the Feynman-Hellman theorem.

b) Use the Feynman-Hellman theorem to calculate the expectation values of $\langle \hat{x}^2 \rangle$ and $\langle \hat{p}^2 \rangle$ for the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 - k\hat{x}$$

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c) The first order correction to the hydrogen spectrum due to fine structure splitting is given by

$$\Delta E_{n,j} = E_n \left(\frac{Z\alpha}{n}\right)^2 \left(\frac{n}{j+1/2} - \frac{3}{4}\right)$$

Using the selection rules $\Delta l = \pm 1$ and $\Delta j = 0, \pm 1$ determine the fine structure splitting of the Balmer H_α line (you may ignore the Lamb shift). *By splitting, I mean the number of lines, as well as the shifts in terms of the wavenumber from the Bohr model result.* $[3+4+3]$

Q 4) a)The Hamiltonian for a system is given by

$$H = \alpha t J_3 + \beta \cos t\, J_1$$

where α and β are constants and J_i are angular momentum generators. Let this be split into $H_0 \equiv \alpha t J_3$ and $H' = \beta \cos t\, J_1$. Write down the interaction picture Hamiltonian.

b) Consider the perturbed harmonic oscillator Hamiltonian

$$H = \hbar\omega\left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right) + \alpha\left(\hat{a} + \hat{a}^\dagger\right)^2$$

Calculate the first and second order corrections to the unperturbed ground state energy $\frac{1}{2}\hbar\omega$.$[4+6]$