MID-SEM

MA3202

Total score= min{marks obtained, 40}

(1) a)Find the curvature and torsion of the circular helix:

$$\gamma(\theta) = (a\cos\theta, a\sin\theta, b\theta), \ \theta \in \mathbb{R}.$$

where a and b are constants.

b) Describe all curves in $\mathbb{R}^3$ which have constant curvature and constant torsion. 3+3=6

(2) The simplest type of singular point of a curve γ is an ordinary cusp: a point p of γ, corresponding to a parameter value t_0, say, is an ordinary cusp if $\gamma'(t_0) = 0$ and the vectors $\gamma''(t_0)$ and $\gamma'''(t_0)$ are linearly independent (in particular, these vectors must both be non-zero). Show that:
i)The curve $\gamma(t) = (t^m, t^n)$, where m and n are positive integers, has an ordinary cusp at the origin if and only if $(m, n) = (2, 3)$ or $(3, 2)$.

ii) If γ has an ordinary cusp at a point p, so does any reparametrization of γ. 3+3=6

(3) i) Show that, if γ is a unit-speed curve,

$$\dot{\mathbf{n}}_s = -\kappa_s \mathbf{t}.$$

$\mathbf{n}_s$ and κ_s stand for signed unit normal and signed curvature respectively.
ii) Let $\gamma(t)$ be a regular plane curve and let λ be a constant. The parallel curve γ^λ of γ is defined by

$$\gamma^\lambda(t) = \gamma(t) + \lambda \mathbf{n}_s(t).$$

Show that, if $\lambda\kappa_s(t) \neq 1$ for all values of t, then γ^λ is a regular curve and that its signed curvature is $\frac{\kappa_s}{|1-\lambda\kappa_s|}$. 2+4=6

(4) Let $\gamma : (\alpha, \beta) \to \mathbb{R}^3$ be a unit-speed curve. If the trace (image) of γ is included in a sphere and it has constant torsion τ_0, prove that there exists b, c such that

$$\kappa(s) = \frac{1}{b\, cos(\tau_0 s) + c\, sin(\tau_0 s)},$$

where κ is the curvature of γ. 6

(5) i) Show that the following is a smooth surface

$$x^2 + y^2 + z^4 = 1.$$

ii) Let γ be a unit-speed curve in $\mathbb{R}^3$ with nowhere vanishing curvature. The tube of radius $a > 0$ around γ is the surface parametrized by

$$\sigma(s, \theta) = \gamma(s) + a(\mathbf{n}(s)\cos\theta + \mathbf{b}(s)\sin\theta),$$

where n is the principal normal of γ and b is its binormal. Prove that σ is regular if the curvature κ of γ is less than a^{-1} everywhere. 2+4=6

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(6) i) Let $\mathcal{S} = \{p \in \mathbb{R}^3 : |p|^2 - \langle p, a\rangle^2 = r^2\}$, with $|a| = 1$, $r > 0$, be a right cylinder of radius r whose axis is the line passing through the origin with direction a. Prove that

$$T_p\mathcal{S} = \{v \in \mathbb{R}^3 : \langle p, v\rangle - \langle p, a\rangle\langle a, v\rangle = 0\}.$$

ii) Let $f : \mathcal{S}_1 \to \mathcal{S}_2$ be a local diffeomorphism and let γ be a regular curve on $\mathcal{S}_1$. Show that $f \circ \gamma$ is a regular curve on $\mathcal{S}_2$. 4+2=6

(7) Let $f : \mathcal{S}_1 \to \mathcal{S}_2$ be a differentiable map between surfaces. If $p \in \mathcal{S}_1$ and $\{e_1, e_2\}$ is an orthonormal basis of $T_p\mathcal{S}$, we define the absolute value of the Jacobian of f at p as

$$|Jac\ f|(p) = |D_P f_1(e_1) \times D_p f_2(e_2)|.$$

i) Prove that this definition does not depend on the chosen orthonormal basis.
ii) Prove that $|Jac\ f|(p) \neq 0$ if and only if f is a local diffeomorphism. 7+3=10