MATH 211Assignment 1

Q 1) Calculate the following :

(i) $\dfrac{(1+i)^3}{(1-i)^5}$ (ii) $\left(1-i\sqrt{3}\right)^3\left(\sqrt{3}+i\right)^2$ (iii) $\left(\sqrt{3}+i\right)^6$

Q 2) (a) Prove De Moivre's formula

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

by induction on n.

(b) Use De Moivre's formula to derive an expression for (a) $\cos 4\theta$ and (b) $\sin 4\theta$.

Q 3) Find all roots for (a) $\sqrt[3]{1-i}$, (b) $\sqrt[4]{16}$ and (c) $\sqrt[8]{i}$.

Q 4) (a) If $z \neq 1$ show that

$$1 + z + z^2 + \ldots + z^n = \frac{1-z^{n+1}}{1-z}.$$

What is the value of the sum if $z = 1$?

(b) Use $z = e^{i\theta}$ above to calculate the sums

$$\sum_{k=0}^{n} \cos(k\theta) \qquad \text{and} \qquad \sum_{k=0}^{n} \sin(k\theta).$$

Q 5) For the function $f(z) = z^2 + z^{-2} - 5$ calculate $f(-1+i)$ and $f\left(1+i\sqrt{3}\right)$.

Q 6) Express the function $f(z) = \dfrac{z+2-i}{z-1+i}$ in the Cartesian form $u(x,y) + iv(x,y)$.

Q 7) Let $f(z) = r^2\cos(2\theta) + ir^2\sin(2\theta)$, where $z = re^{i\theta}$. Express $f(z)$ in the form $u(x,y) + iv(x,y)$.

Q 8) Find the image in the w plane of the open disk $\{z \in \mathbb{C} : |z+1+i| < 1\}$ under the transformation $z \mapsto w = (3-4i)z + 6 + 2i$.

Q 9) Find the image in the w plane of the right half plane $\{z \in \mathbb{C} : \text{Re}(z) > 1\}$ under the transformation $z \mapsto w = (-1+i)z - 2 + 3i$.

Q 10) What does the vertical line $x =$ constant map into under the action of $f(z) = z^2$? What about the horizontal line $y =$ constant? Show that the two resulting curves above intersect each other at right angles.

Q 11) Sketch the set of points satisfying (a) $\text{Re}\left(z^2\right) > 4$, and (b) $\text{Im}\left(z^2\right) > 6$.

Q 12) If $u(x,y) = \dfrac{x^2+y^4}{x^2+y^2}$, then show that the limit $\lim_{(x,y)\to(0,0)} u(x,y)$ does not exist. What about the limit $\lim_{(x,y)\to(0,0)} v(x,y)$ where $v(x,y) = \dfrac{x^3+y^3}{x^2+y^2}$?

Q 13) Let $f(z) = \dfrac{z^2}{|z|^2}$. Find

(a) $\lim_{z\to 0} f(z)$ as z approaches 0 along the x axis.

(b) $\lim_{z\to 0} f(z)$ as z approaches 0 along the y axis.

(c) $\lim_{z\to 0} f(z)$ as z approaches 0 along the line $y = x$.

(d) $\lim_{z\to 0} f(z)$ as z approaches 0 along the line $y = 2x$.

(e) $\lim_{z\to 0} f(z)$ as z approaches 0 along the parabola $y = x^2$.

What conclusion can you draw about $\lim_{z\to 0} f(z)$ from these results?

Q 14) Where are the following functions continuous?

(a) $z^4 - 9z^2 + iz - 2$ (b) $\dfrac{z+1}{z^3+1}$ (iii) $\dfrac{z^2+1}{z^4+2z^2+1}$ (iv) $\dfrac{x+iy}{x-y}$ (v) $\dfrac{x+iy}{x-iy}$.

Q 15) Let $f(z)$ be continuous for all values of $z \in \mathbb{C}$. Show that the functions $g(z)$ and $h(z)$ defined by $g(z) = f(\overline{z})$ and $h(z) = \overline{f(z)}$ are continuous for all values of z.

Q 16) Show using the Cauchy-Riemann conditions (and the corresponding sufficiency criteria) that the following functions are differentiable for all $z = x + iy$, and find $f'(z)$

(i) $f(z) = iz^3 + 4i$
(ii) $f(z) = e^{z^2}$
(iii) $f(z) = -2(xy + x) + i(x^2 - 2y - y^2)$
(iv) $f(z) = e^{2xy}\left[\cos(y^2 - x^2) + i\sin(y^2 - x^2)\right]$

Q 17) Find the real constants a and b such that the function $f(z) = (2x - y) + i(ax + by)$ is entire. What can you say about a and b if you allow them to be complex.

Q 18) The polar form of the Cauchy-Riemann conditions proceeds by writing the function $f : \mathbb{C} \to \mathbb{C}$ in the form

$$f(z) = u(r,\theta) + iv(r,\theta),$$

where $z = re^{i\theta}$. By demanding that for a function to be differentiable at $z_0 \equiv r_0 e^{i\theta_0}$ the limits $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ must be the same as z approaches z_0 along a line of constant $r = r_0$ and along a line of constant $\theta = \theta_0$, derive the polar version of the Cauchy-Riemann conditions.

Use this condition to prove that the function $f(z) = z^4$ is differentiable everywhere.

Q 19) Derive the polar form of the Cauchy-Riemann conditions by starting from the Cartesian version, and carrying out the change of variables :

$$x = r\cos\theta, \qquad y = r\sin\theta$$

using the chain rules $u_r = u_x \frac{\partial x}{\partial r} + u_y \frac{\partial y}{\partial r}, \ldots$.

Q 20) Let the function $f(z)$ be holomorphic in a domain D. If its modulus $|f(z)|$ is constant in D, then show that $f(z)$ is constant in D. *Hint : Use the fact that* $uu_x + vv_x = uu_y + vv_y = 0$ *(why?) and the Cauchy-Riemann conditions.*

Q 21) Where is the function

$$f(z) = (x^3 + 3xy^2) + i(y^3 + 3x^2y)$$

differentiable? Where is it analytic?

Q 22) Let a, b, c be real constants. Determine a relation between them so that the function $u(x,y) = ax^2 + bxy + cy^2$ is harmonic.

Q 23) Find the analytic function $f(z) = u(x,y) + iv(x,y)$ given the following :

(i) $u(x,y) = y^3 - 3x^2y$ (ii) $u(x,y) = \sin y \sinh x$ (iii) $v(x,y) = e^y \sin x$ (iv) $v(x,y) = \sin x \cosh y$

Q 24) Let $f(z)$ and $\overline{f(z)}$ be holomorphic functions of z over a domain D. Show that $f(z)$ must be a constant in D.

Q 25) If u is a harmonic function in a domain D and $u + iv$ is holomorphic in D, then we call v the harmonic conjugate of u. Let v be the harmonic conjugate of u.
(a) Find the harmonic conjugate of v.
(b) Show that $h = v^2 - u^2$ is a harmonic function.