MATH211 Assignment 2

Q 1) Where are the following functions (i) differentiable, (ii) holomorphic?
(a) $f(z) = \dfrac{z^2}{z^3 + a^3}$,
(b) $f(z) \equiv f(x+iy) = e^{x^4 - 6x^2y^2 + y^4}\left[\cos\left(4x^3y - 4xy^3\right) - i\sin\left(4x^3y - 4xy^3\right)\right]$
(c) $f(z) \equiv f(x+iy) = 8x - x^3 - xy^2 + i\left(x^2y + y^3 - 8y\right)$.

Q 2) Which of the following functions are harmonic? For the ones that are, find the harmonic conjugate.
(a) $u(x,y) = x^3 - 3x^2y$, (b) $u(x,y) = x^3 - 3xy^2$, (c) $u(x,y) = \cos x \sinh y$,
(d) $u(x,y) = \sin x \sinh y$

Q 3) Sketch the image of the square bounded by the points $0, 1, 1+i, i$ under the map $z \to z^2$.

Q 4) Show that under any holomorphic map, the images of the real and imaginary axis intersect at right angles.

Q 5) Identify the location and order of the branch point singularities for each of the following multifunctions. Sketch appropriate branch cuts for them:
(a) $f(z) = z^{3.2}(z-1)$, (b) $f(z) = (z-i)^{\pi}(z+i)^{-\pi}$, (c) $f(z) = \frac{\log z}{z+2}$,
(d) $f(z) = z^{1/2}(1-z)^{2/3}$

Q 6) Let $f(z) \equiv f\left(re^{i\theta}\right) = u(r,\theta) + iv(r,\theta)$ be analytic in a domain D which does not contain the origin. Use the polar form of the Cauchy-Riemann conditions $u_\theta = -rv_r$, $v_\theta = ru_r$ to show that

$$r^2 u_{rr} + ru_r + u_{\theta\theta} = 0.$$

By substituting $x = r\cos\theta$, $y = r\sin\theta$ rewrite the above equation as a partial differential equation satisfied by the function $U(x,y)$ where $U(x,y) = u(r,\theta)$.

Q 7) Show that the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(n+1+i)(n+i)}$ converges and calculate its limit. *Hint : It may be a good idea to rewrite the summand in terms of partial fractions.*

Q 8) Find the disc of convergence of the following series :
(a) $\displaystyle\sum_{n=99}^{\infty} (1+i)^n z^n$, (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(z-i+1)^n}{(3+4i)^n}$, (c) $\displaystyle\sum_{n=0}^{\infty} n! z^n$,
(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2}\left(\frac{z}{2}\right)^{2n}$

Q 9) The power series function defined in part (d) of the last question defines the Bessel function of order 0, $J_0(z)$. By differentiating term by term, show that this satisfies

$$zJ_0''(z) + J_0'(z) + zJ_0(z) = 0$$

for all $z \in \mathbb{C}$.

Q 10) It is known that the power series $\sum_{n=0}^{\infty} c_n (z - z_0)^n$ converges at the origin, while it diverges at the points $z = -4i$, $z = 3 + 2i$ and $z = -2 - i$. On an Argand diagram sketch the region in which the center of the series z_0 must be located.