MATH211 Assignment 3

Q 1) Use the power series method to find two linearly independent solutions to the equation

$$e^x \frac{d^2y}{dx^2} + x\frac{dy}{dx} + x^2 y = 0.$$

Note that the point $x = 0$ is a regular point of the equation and choose a power series solution of the form

$$y(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

Remember to expand the coefficient e^x as a power series in x as well! Find at least the first 4 terms in each independent solution.

Q 2) Find the singular points of each of the following differential equations and determine the nature of the singularities. (Remember to consider the point $z = \infty$!) :

(a) $\left(1 - z^2\right) \frac{d^2y}{dz^2} - z\frac{dy}{dz} + n^2 y = 0, \qquad n \in \mathbb{Z}.$

(b) $z\frac{d^2y}{dz^2} + (1 - z)\frac{dy}{dz} + \alpha y = 0, \qquad \alpha = \text{constant}.$

(c) $z(1 - z)\frac{d^2y}{dz^2} + [\gamma - (1 + \alpha + \beta)z]\frac{dy}{dz} - \alpha\beta y = 0, \qquad \alpha, \beta, \gamma = \text{constants}.$

(d) $z\frac{d^2y}{dz^2} + (\gamma - z)\frac{dy}{dz} - \alpha y = 0, \qquad \alpha, \gamma = \text{constants}.$

Q 3) Solve equations 2c (the hypergeometric equation) and 2d (the confluent hypergeometric equation) above using the Frobenius method, assuming that $\gamma \notin \mathbb{Z}$ (can you see the significance of this condition?).

Q 4) Find the general solution to the differential equations :

$$z(1 - z)\frac{d^2y}{dz^2} + [1 - 3z]\frac{dy}{dz} - y = 0, \qquad \alpha, \beta, \gamma = \text{constants}$$

and

$$z(1 - z)\frac{d^2y}{dz^2} + [3 - 3z]\frac{dy}{dz} - y = 0, \qquad \alpha, \beta, \gamma = \text{constants}$$

Note that both of these equations are special cases of the hypergeometric equation, but this time with $\gamma \in \mathbb{Z}$.

Q 5) One solution of the equation

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + \left(z^2 - 1\right) y = 0$$

is given by

$$J_1(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+1)!} \left(\frac{z}{2}\right)^{2n+1}$$

Use both of the two methods we have discussed in the class to find out the second solution to this equation (determine at least the first four terms). Are the solutions that you obtain by the two methods identical? Explain.

Q 6) Show that the point $z = 0$ is a regular singular point of the equation

$$z^2 \frac{d^2y}{dz^2} + \sin z \frac{dy}{dz} - \cos z\, y = 0.$$

Use the Frobenius method to find two linearly independent solutions of this equation. Repeat for the equation

$$z^2 \frac{d^2y}{dz^2} + \sin z \frac{dy}{dz} + \cos z\, y = 0.$$

Q 7) Show that the Legendre equation

$$\left(1 - z^2\right) \frac{d^2y}{dz^2} - 2z\frac{dy}{dz} + l\,(l+1)\,y = 0 \tag{1}$$

has a regular singular point at $z = \infty$. This means that it has a solution of the form

$$y(z) = \sum_{n=0}^{\infty} c_n \left(\frac{1}{z}\right)^{n+s}$$

Obtain a solution of this form (find at least the first four coefficients).

Q 8) (Rodriguez formula for Legendre polynomials) Remember that for non-negative integer l, the Legendre equation

$$\left(1 - z^2\right) \frac{d^2y}{dz^2} - 2z\frac{dy}{dz} + l\,(l+1)\,y = 0$$

has one solution that is well behaved at the singular points $z = \pm 1$ (the other solution will be singular there). This solution, as we have seen in class, is a polynomial of degree l. We define *the* Legedre polynomial of degree l to be polynomial, denoted $P_l(z)$ that satisfies this equation and is normalised by $P_l(1) = 1$. We have learnt how to find them by using the recursion relations that follow from the power series method. In this exercise we will learn an alternative expression for these polynomials.

Start with the function $F(z) = \left(z^2 - 1\right)^l$.

(i) Show that this obeys

$$\left(z^2 - 1\right) \frac{dF}{dz} = 2lzF$$

(ii) Differentiate both sides $l+1$ times successively with respect to z to show that the function $\frac{d^l F}{dz^l}$ satisfies the Legendre equation. *Hint : Use Liebnitz formula for the successive differentiation of a product :*

$$\frac{d^n}{dz^n}(uv) = \sum_{r=0}^{n} \binom{n}{r} \frac{d^r u}{dz^r} \frac{d^{n-r} v}{dz^{n-r}}$$

(iii) Argue that $\frac{d^l F}{dz^l}$ is a polynomial of degree l - and so it is, up to a multiplicative constant, the polynomial $P_l(z)$.

(iv) By writing $\left(z^2-1\right)^l = (z-1)^l (z+1)^l$ and using the Liebnitz rule, show that

$$\left.\frac{d^l F}{dz^l}\right|_{z=1} = 2^l l!$$

Hence show that we must have

$$P_l(z) = \frac{1}{2^l l!} \frac{d^l}{dz^l} \left(z^2-1\right)^l . \tag{2}$$

This is called the Rodriguez formula for the Legendre polynomial.

(v) Use the Rodriguez formula to calculate the first six Legendre polynomials.

(vi) Use the Rodriguez formula to calculate $P_l(-1)$ and $P_l(0)$.

(vii) Use the Rodriguez formula to calculate the values for

$$\int_{-1}^{+1} P_l(z)\, dz \qquad \text{and} \qquad \int_0^1 P_l(z)\, dz.$$

Q 9) The equation

$$\left(1-z^2\right)\frac{d^2y}{dz^2} - 2z\frac{dy}{dz} + \left[l(l+1) - \frac{\mu^2}{1-z^2}\right] y = 0 \tag{3}$$

is called the associated Legendre equation where $l, \mu > 0$ are constants.

(a) Show that the leading behavior of the solutions to this equation as one approached the singular points $z = \pm 1$ is $(1-z)^{\pm\mu/2}$ and $(1+z)^{\pm\mu/2}$, respectively (*Hint: substitute $u = 1-z$ to find the leading behavior as $z \to 1$*).

(b) Define a function $f(z)$ by $y(z) = \left(1-z^2\right)^{\mu/2} f(z)$. Find the differential equation satisfied by the function $f(z)$.

(c) From this point on, we will consider the case where l, μ are both non-negative integers. Differentiate the Legendre equation μ times with respect to z to show that one solution for the equation found in part (b) above is satisfied by

$$\frac{d^\mu}{dz^\mu} P_l(z)$$

so that a solution of the associated Legendre equation is given by

$$P_l^{\mu}(z) \equiv (-1)^{\mu} \left(1 - z^2\right)^{\mu/2} \frac{d^{\mu}}{dz^{\mu}} P_l(z) . \tag{4}$$

Q 10) (a) Solve the equation that you obtained in 9b above using the power series method by expanding $f(z)$ in a power series centered at the regular point $z = 0$. At this point allow l, μ to be any constants (not necessarily integers)

(b) Argue that as $z \to \pm 1$, the power series you obtain behaves like $\left(1 - z^2\right)^{-\mu}$ so that the original $y(z)$ behaves like $\left(1 - z^2\right)^{-\mu/2}$.

(c) What restriction must you place on the constants l and μ to ensure that a solution does not blow up as $z \to \pm 1$?

(d) Since $z = 0$ is a regular point of the associated Legendre equation (3), you could have applied the power series method to this equation directly. Show that this leads to a three term recursion relation and use it to calculate the first few terms of the solution.