MA211 Mid-Semester Examination Solutions
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Q 1) Use De Moivre's theorem to derive an expression for $\cos(4\theta)$ in terms of $\cos\theta$ and $\sin\theta$.

Ans : De Moivre's theorem :

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Thus

$$\begin{aligned}
\cos 4\theta + i\sin 4\theta &= (\cos\theta + i\sin\theta)^4 \\
&= (\cos\theta)^4 + 4(\cos\theta)^3(i\sin\theta) + 6(\cos\theta)^2(i\sin\theta)^2 \\
&\quad +4(\cos\theta)(i\sin\theta)^3 + (i\sin\theta)^4
\end{aligned}$$

Taking the real part

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

Q 2) Consider the map $z \mapsto z^{1/4}(1-z)^{1/4}$. What are the branch points and what is the order of each? Sketch at least two different branch cuts for this function.

Ans : Branch points z=0, z=1. The order of each is 3.

A branch cut for this function is any pair of lines joining 0 and 1, respectively to the point at ∞.

Q 3) Deduce an explicit formula for a Mobius transformation that transforms three points $1, 3+i, 2-i$ to the three points 0,1 and ∞, respectively, of the extended complex plane.

Ans : the Mobius transformation is

$$z \mapsto M(z) = \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{C}$$

$M(1) = 0 \implies a+b=0$, while $M(2-i) = \infty \implies c(2-i)+d = 0$. Thus $b = -a$and $d = (i-2)c$, so that we have

$$M(z) = \frac{a}{c}\frac{z-1}{z+i-2}$$

Finally

$$M(3+i) = 1 \implies \frac{a}{c}\left(\frac{2+i}{1+2i}\right) = 1$$

Thus

$$M(z) = \frac{1+2i}{2+i} \frac{z-1}{z+i-2}$$

Q 4) Show that the function $e^{x^2-y^2}\left(2\cos^2(xy)-1\right)$ is harmonic. Find out its harmonic conjugate.

Ans : Note that the function is

$$u = \Re\left(e^{z^2}\right),$$

thus the real part of a holomorphic function and hence harmonic (no lengthy calculations are necessary. The most general function for which u is the real part is $e^{z^2} + ig(x,y)$ where g is an arbitrary real function of two variables. However, for this to be holomorphic we need $ig(x,y)$ to be holomorphic. The CR equations yield

$$g_x = g_y = 0$$

and thus g is a constant. Thus, the most general harmonic conjugate for u is

$$v = \Im\left(e^{z^2} + ic\right) = e^{x^2-y^2}\sin(2xy) + c$$

Q 5) Prove that the limit of a complex sequence is unique.

Ans : See the notes for lecture 11.