Complex numbers - History

Ananda Dasgupta

MA211, Lecture 1

The genesis of complex numbers

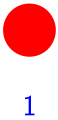
A fantasy!

In the beginning was counting!

The genesis of complex numbers

A fantasy!

In the beginning was counting!



The genesis of complex numbers

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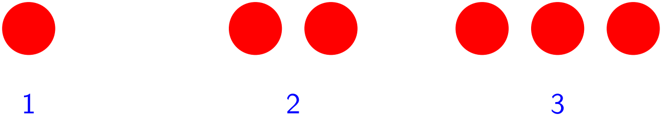

The genesis of complex numbers

A fantasy!

Natural numbers

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

The genesis of complex numbers

A fantasy!

We can now write equations using natural numbers only

$$x + 3 = 5$$

The genesis of complex numbers

A fantasy!

We can now write equations using natural numbers only

$$x + 3 = 5 \qquad \Rightarrow \qquad x = 2$$

The genesis of complex numbers

A fantasy!

We can now write equations using natural numbers only

Not all such equations can be solved within $\mathbb{N}$!

$$x + 5 = 5$$

The genesis of complex numbers

A fantasy!

We can now write equations using natural numbers only
Not all such equations can be solved within $\mathbb{N}$!

$$x + 5 = 5$$

$$x + 5 = 3$$

The genesis of complex numbers
A fantasy!

We need to expand the set $\mathbb{N}$ to the set $\mathbb{Z}$

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

The genesis of complex numbers

A fantasy!

All equations involving addition can be solved within $\mathbb{Z}$

$$x + 5 = 5 \qquad \Rightarrow \qquad x = 0$$
$$x + 5 = 3 \qquad \Rightarrow \qquad x = -2$$

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A fantasy!

All equations involving addition can be solved within $\mathbb{Z}$
This is not true, however, for equations involving multipliction!

$$5x = 3$$

The genesis of complex numbers

A fantasy!

We need to expand the set $\mathbb{Z}$ to the set $\mathbb{Q}$

$$\mathbb{Q} = \left\{ \frac{p}{q} \ : \ p, q \in \mathbb{Z} \right\}$$

The genesis of complex numbers

A fantasy!

We need to expand the set $\mathbb{Z}$ to the set $\mathbb{Q}$

$$5x = 3 \qquad \Rightarrow \qquad x = \frac{3}{5}$$

The genesis of complex numbers

A fantasy!

Even $\mathbb{Q}$ is not algebraically complete.

$$x^2 - 2 = 0$$

The genesis of complex numbers

A fantasy!

Even $\mathbb{Q}$ is not algebraically complete.
Algebraic completion of $\mathbb{Q}$ leads us to $\mathbb{R}$.

$$x^2 - 2 = 0 \qquad \Rightarrow \qquad x = \sqrt{2}$$

The genesis of complex numbers

A fantasy!

You can write algebraic equations involving real numbers only that can not be solved in real numbers!

$$x^2 + 1 = 0$$

The genesis of complex numbers

A fantasy!

We have to enlarge the set to $\mathbb{C}$

$$x^2 + 1 = 0 \qquad \Rightarrow \qquad x = \mathrm{i}$$

The genesis of complex numbers
A fantasy!

The sequence ends here!

The genesis of complex numbers

A fantasy!

The sequence ends here!
All polynomial equation in $\mathbb{C}$

$$c_n z^n + c_{n-1} z^{n-1} + \ldots + c_1 z + c_0 = 0$$

The genesis of complex numbers

A fantasy!

The sequence ends here!
All polynomial equation in $\mathbb{C}$ has a solution in $\mathbb{C}$!

$$\exists z \in \mathbb{C} \ : \ c_n z^n + c_{n-1} z^{n-1} + \ldots + c_1 z + c_0 = 0$$

The genesis of complex numbers

A fantasy!

The sequence ends here!
All polynomial equation in $\mathbb{C}$ has a solution in $\mathbb{C}$!
The fundamental theorem of algebra

$$\exists z \in \mathbb{C} \ : \ c_n z^n + c_{n-1} z^{n-1} + \ldots + c_1 z + c_0 = 0$$

The genesis of complex numbers

A more historical account!


HIERONYMI CAR
DANI, PRÆSTANTISSIMI MATHE
MATICI, PHILOSOPHI, AC MEDICI,
ARTIS MAGNÆ,
SIVE DE REGVLIS ALGEBRAICIS,
Lib. unus. Qui & totius operis de Arithmetica, quod
OPVS PERFECTVM
inscriptit, est in ordine Decimus.

Habes in hoc libro, studiose Lector, Regulas Algebraicas (Itali, de la Cossa uocant) nouis adinuentionibus, ac demonstrationibus ab Authore ita locupletatas, ut pro pauculis antea uulgò tritis, iam septuaginta euaserint. Neque solum, ubi unus numerus alteri, aut duo uni, uerum etiam, ubi duo duobus, aut tres uni equales fuerint, nodum explicant. Hunc aũt librum ideo seorsim edere placuit, ut hoc abstrusissimo, & planè inexhausto totius Arithmeticae thesauro in lucem eruto, & quasi in theatro quodam omnibus ad spectandum exposito, Lectores incitarentur, ut reliquos Operis Perfecti libros, qui per Tomos edentur, tanto auidius amplectantur, ac minore fastidio perdiscant.

- Girolomo Cardano's *Ars Magna* (1545) is the birthplace of complex numbers.

The genesis of complex numbers

A more historical account!

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- Solves $x^2 = mx + c$.
- Solution is $x = \frac{1}{2}\left[m \pm \sqrt{m^2 + 4c}\right]$.

The genesis of complex numbers

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- "Subtle as they are useless"!

The genesis of complex numbers

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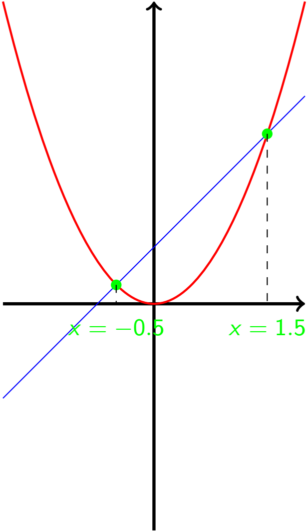
- "Subtle as they are useless"!
- To Cardano, in such cases, the equation has **no** solutions!

The genesis of complex numbers

A more historical account!



To Cardano, as with the ancient Greeks, the equation $x^2 = mx + c$ signified the geometrical problem of finding the points where straight line $y = mx + c$ intersects the parabola $y = x^2$.

The genesis of complex numbers

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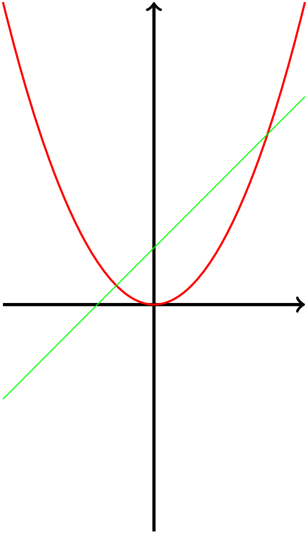

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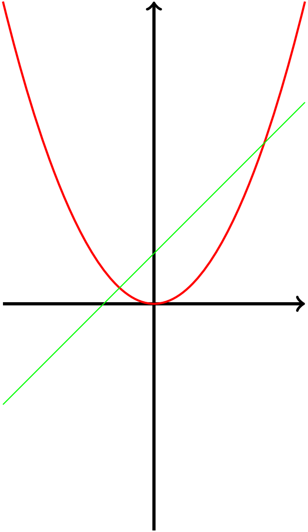

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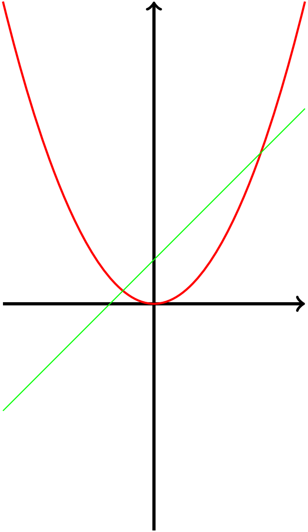

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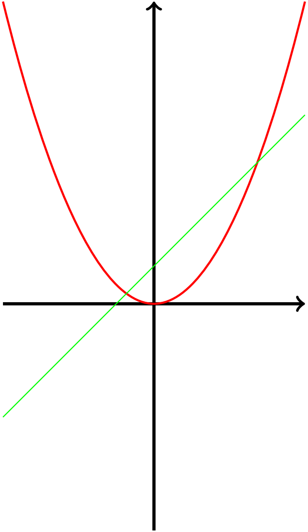

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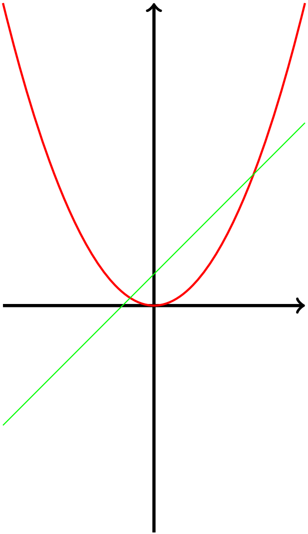

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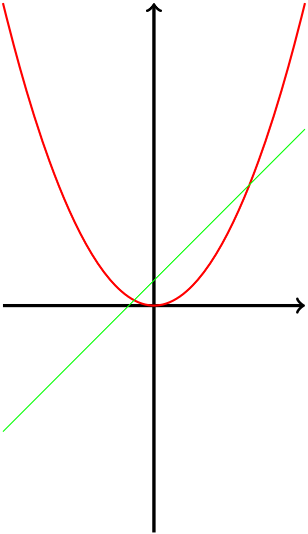

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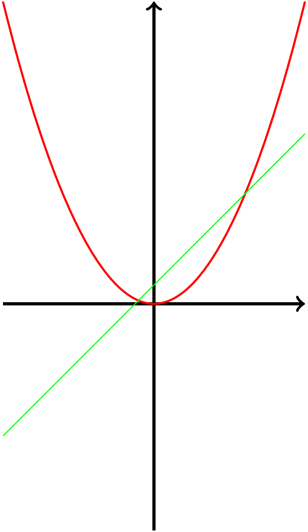

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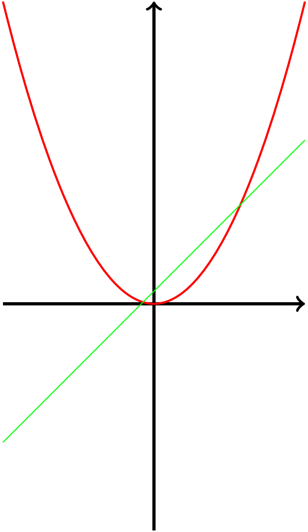

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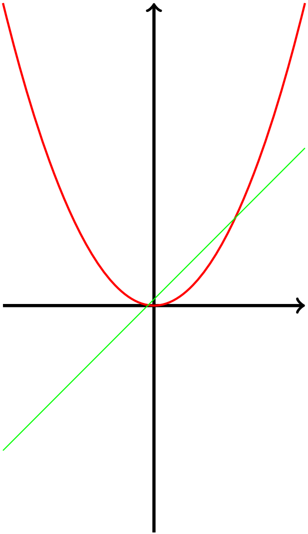

To Cardano, as with the ancient Greeks, the equation $x^2 = mx + c$ signified the geometrical problem of finding the points where straight line $y = mx + c$ intersects the parabola $y = x^2$. For $m^2 + 4c < 0$ there is no intersection!

The genesis of complex numbers

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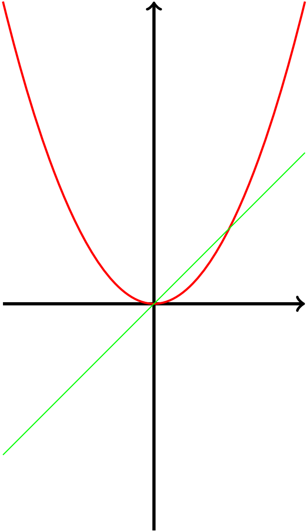

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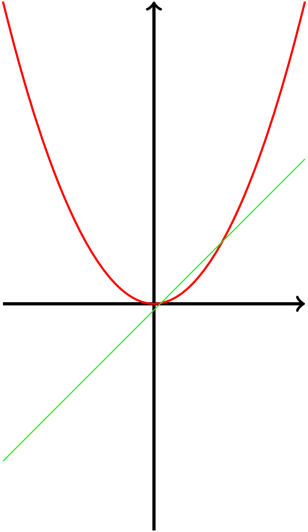

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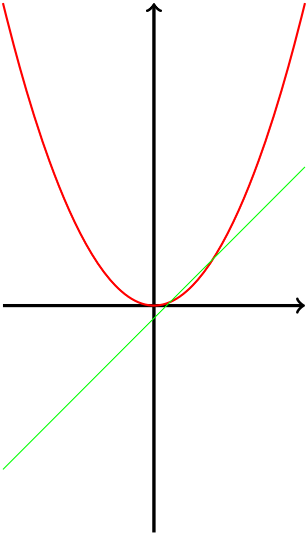

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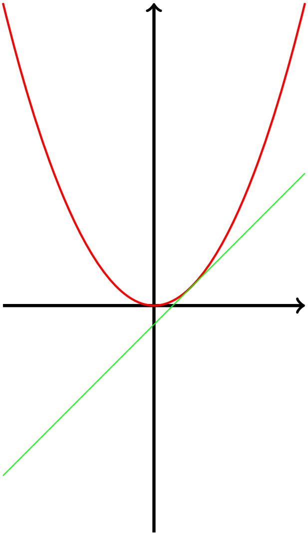

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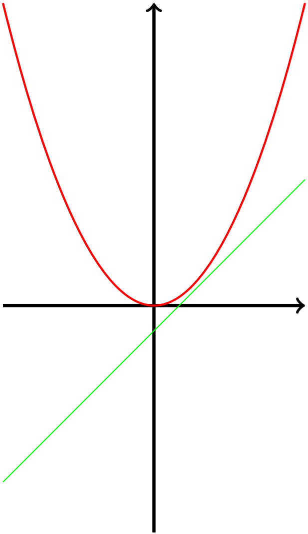

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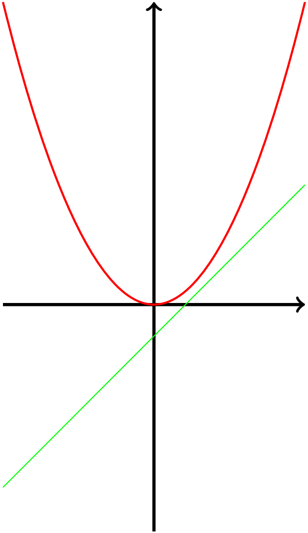

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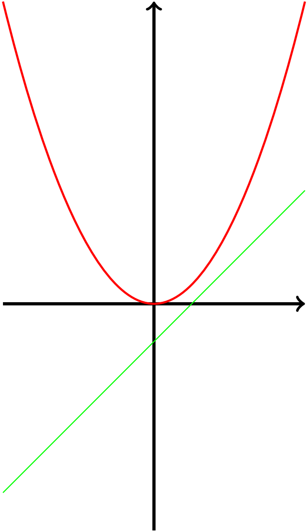

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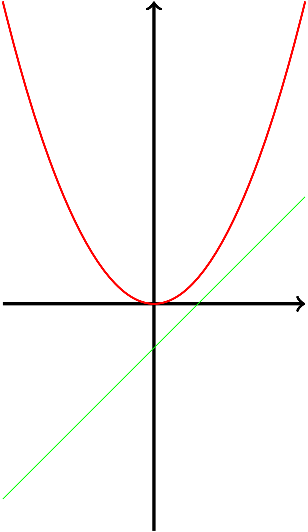

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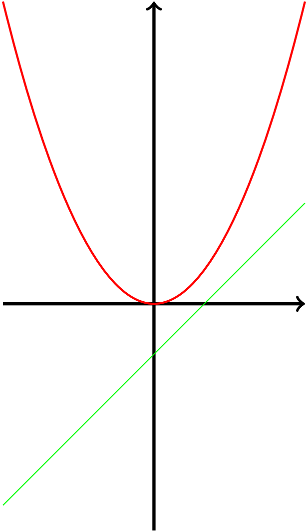

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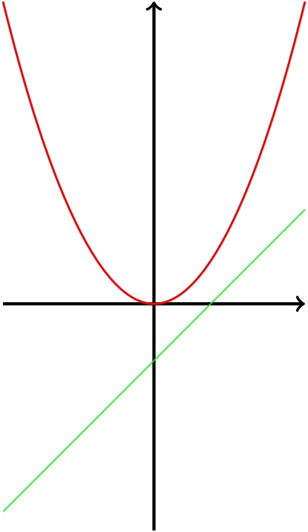

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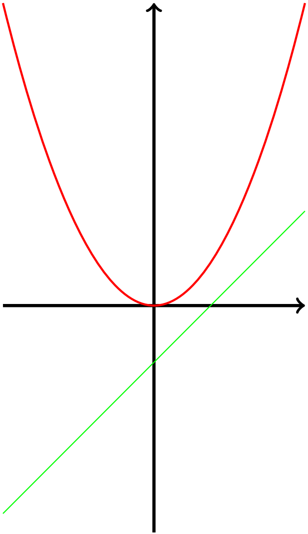

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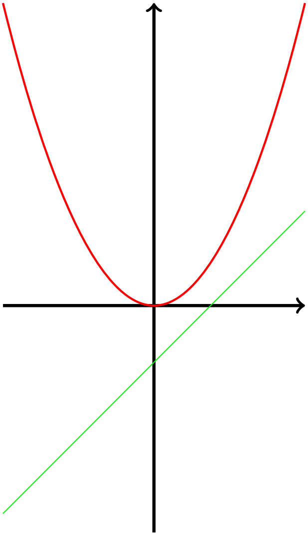

To Cardano, there was no compelling reason to demand that there must be a solution to $x^2 = mx + c$!

The genesis of complex numbers

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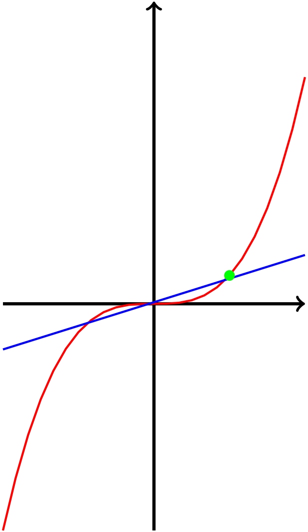

The line $y = 3px + 2q$ always cuts the curve $y = x^3$!

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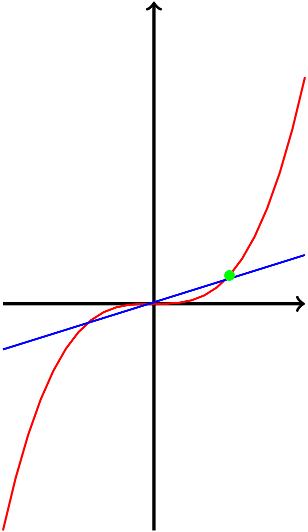

The line $y = 3px + 2q$ always cuts the curve $y = x^3$!
The equation

$$x^3 = 3px + 2q$$

always has a solution!

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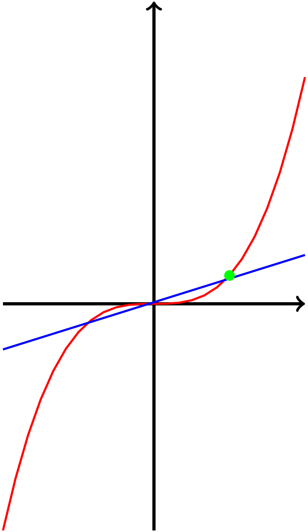

The line $y = 3px + 2q$ always cuts the curve $y = x^3$!
The equation

$$x^3 = 3px + 2q$$

always has a real solution!

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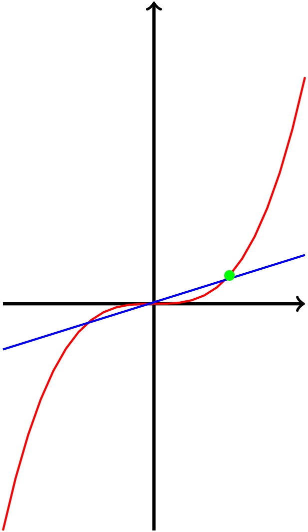

Cardano's book provided the solution

$$x = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}}$$

to

$$x^3 = 3px + 2q$$

based on work done by del Ferro and Tartaglia.

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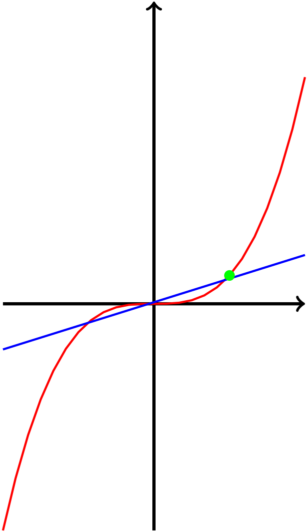

Cardano called the case $q^2 < p^3$ the *casus irreducibilis*.

The genesis of complex numbers

A more historical account!


201

L'ALGEBRA
PARTE MAGGIORE
DELL'ARIMETICA
DI RAFAEL BOMBELLO
BOLOGNESE.

Libro Secondo.

SI marauigliaranno forſe alcuni, che contra l'antico uſo de Scrittori Italiani, i quali ſino à queſto giorno hanno ſcritto di queſta ſcientia dell'Arimetica, quando gli è occorſo di trattare di quantità incognita: eſsi ſempre l'hanno nominata ſotto queſta uoce di (Coſa) come voce commune à tutte le coſe incognite, è d'io chiami hora queſte quantità (Tanti) ma chi bene conſiderarà il fatto, conoſcerà, che più ſe le conuiene queſta uoce di (Tanto), che di (coſa), perche ſe diremo (Tanto) è uoce appropriata à quantità di numeri, ilche non ſi può dire di (coſa) eſſendo quella uoce uniuerſaliſsima, e cõmune ad ogni ſoſtantia coſi ignota come nota. In oltre io trouo, che Diofante Auttor Greco coſi la noma, il ch'è di non pic

O ciolo

Cardano called the case $q^2 < p^3$ the *casus irreducibilis*.
Rafael Bombelli in *L'Algebra* (1572) noted something strange! For

$$x^3 = 15x + 4$$

Cardano's formula yields

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O ciolo

Cardano called the case $q^2 < p^3$ the *casus irreducibilis*.

Rafael Bombelli in *L'Algebra* (1572) noted something strange! For

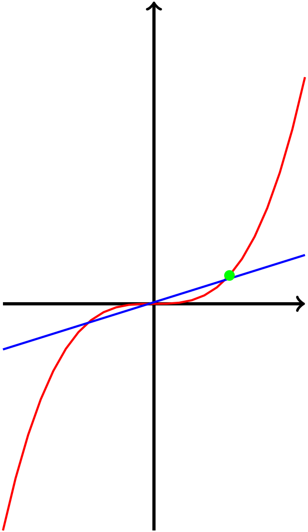
$$x^3 = 15x + 4$$

Cardano's formula yields

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

The genesis of complex numbers

A more historical account!



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But $x = 4$ is an obvious solution!

The genesis of complex numbers

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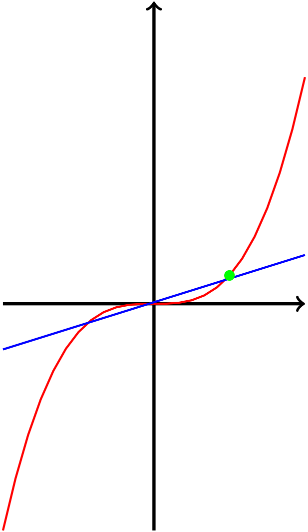

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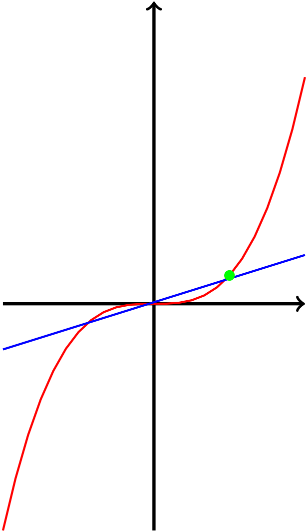

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The genesis of complex numbers

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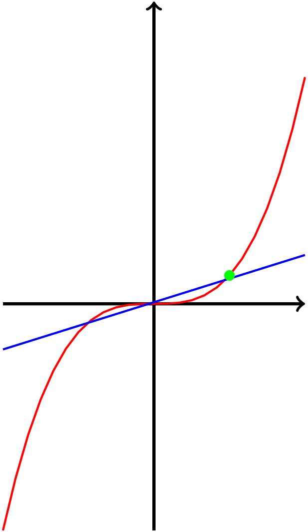

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$$\left(2+\sqrt{-1}\right)^3 =$$

The genesis of complex numbers

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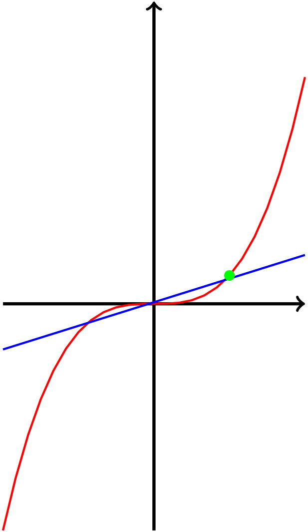

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The genesis of complex numbers

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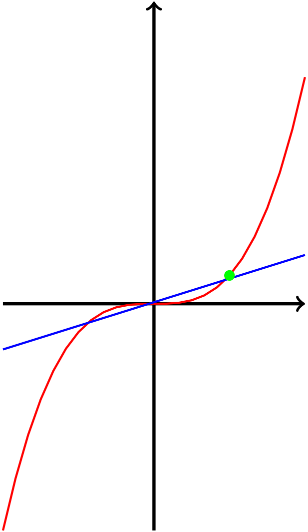

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The genesis of complex numbers

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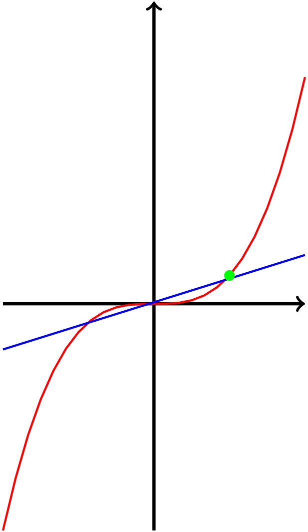

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The genesis of complex numbers

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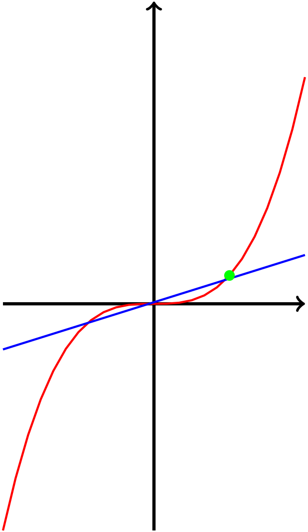
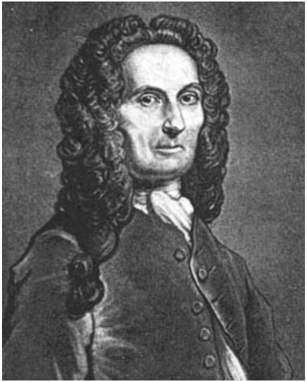


Bombelli's work showed us that complex numbers can be useful, even when dealing with a case where the answer is a real number!

de Moivre and his formula



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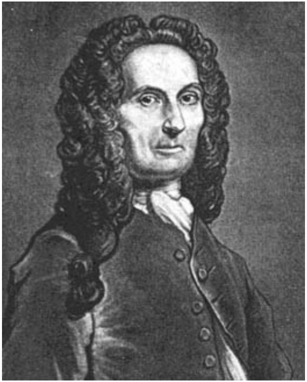

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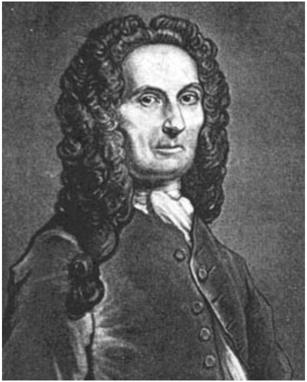

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- In 1698 he wrote down the equivalent to :

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$$

— what is known today as **de Moivre's theorem**.

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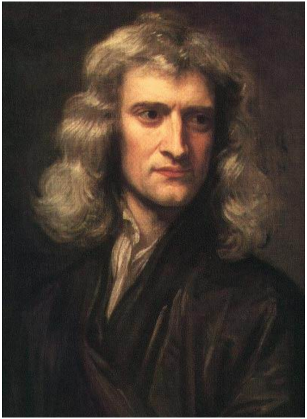

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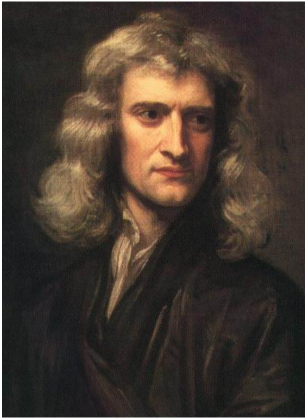

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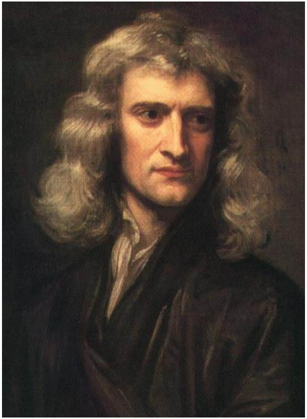

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- Apparently, Newton used this to compute the cube roots that appear in Cardono's formula in his *casus irreducibilis*.
- As early as 1591, François Viète had used an equivalent method!

Wallis : going beyond the real line

Roots of the quadratic

$$x^2 - 2bx + c^2 = 0$$

are

$$-b + \sqrt{b^2 - c^2}$$

and

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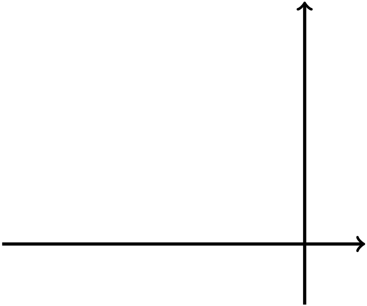
Wallis : going beyond the real line



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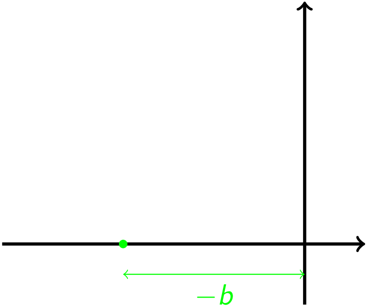

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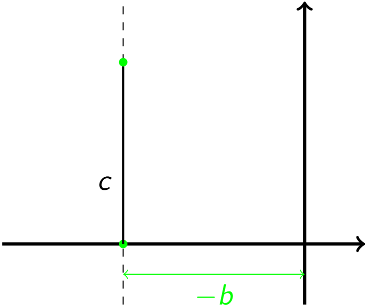

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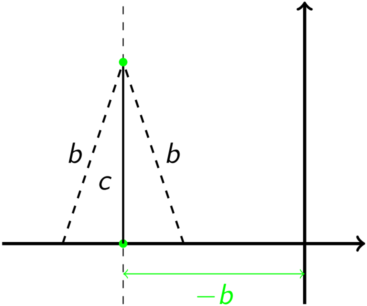

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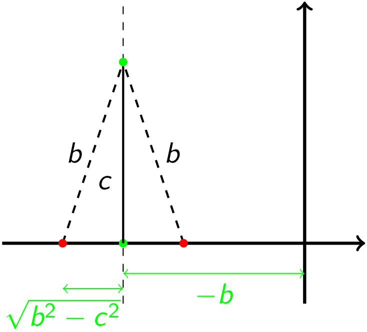

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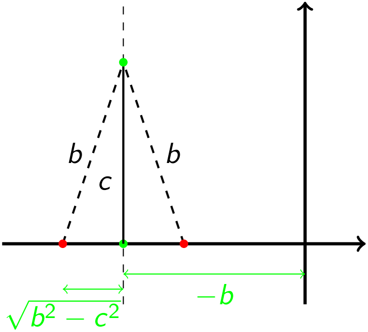

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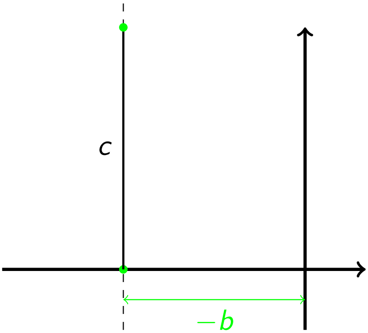

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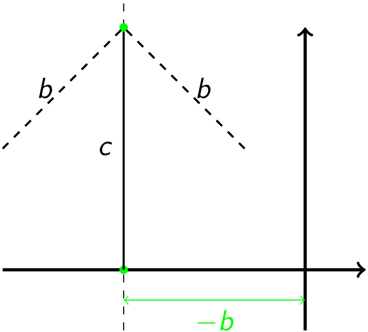

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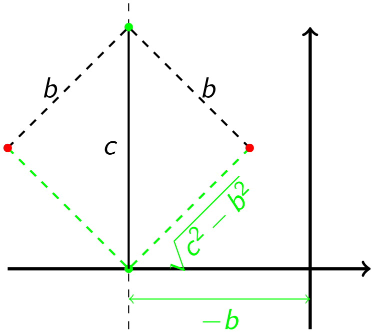

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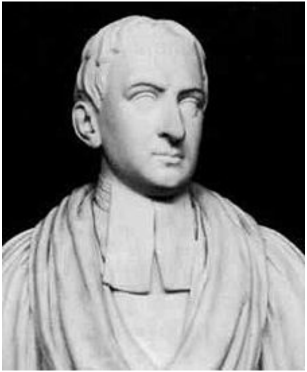
Complex roots move out of the real line - into the plane!

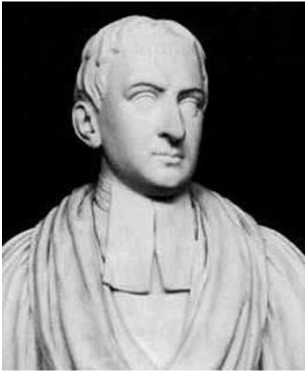
Cotes : the genesis of the complex plane



In 1715, Richard Cotes figured out a way to integrate

$$\int \frac{dx}{x^n - 1}$$

Cotes : the genesis of the complex plane



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His method hinged on factorizing

$$x^n - 1$$

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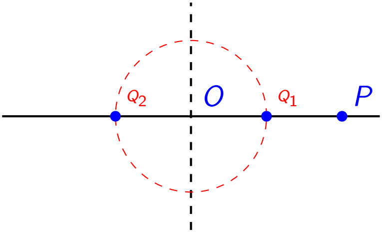
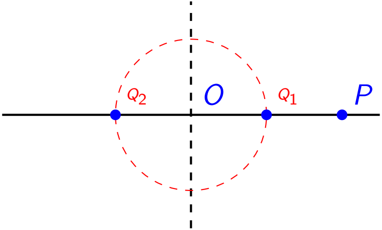


Let's start with $n = 2$:

$$x^2 - 1 = (x - 1)(x + 1)$$

Cotes : the genesis of the complex plane



Let's start with $n = 2$:

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If $\overline{PO} = x$

$$x^2 - 1 = \overline{PQ_1} \cdot \overline{PQ_2}$$

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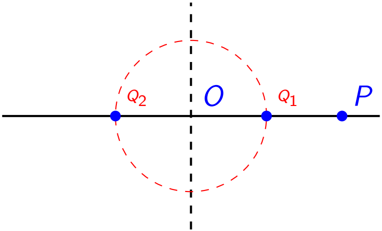
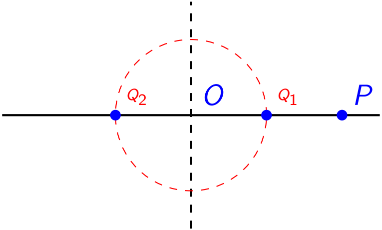


Let's try $n = 4$:

$$x^4 - 1 \quad = \quad (x-1)(x+1)\left(x^2+1\right)$$

Cotes : the genesis of the complex plane



Let's try $n = 4$:

$$\begin{aligned} x^4 - 1 &= (x-1)(x+1)\left(x^2+1\right) \\ &= (x-1)(x+1)\left(\sqrt{x^2+1}\right)^2 \end{aligned}$$

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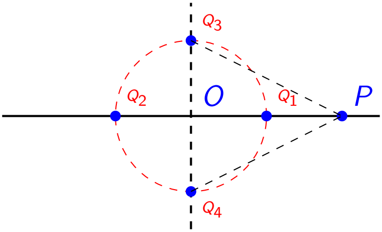

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\begin{aligned}
x^4 - 1 &= (x-1)(x+1)\left(x^2+1\right) \\
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&= \overline{PQ_1} \cdot \overline{PQ_2} \cdot \overline{PQ_3} \cdot \overline{PQ_4}
\end{aligned}
$$

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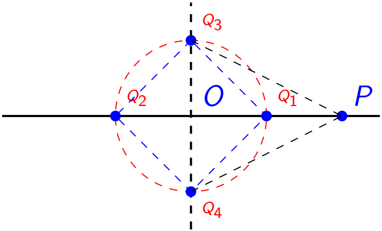

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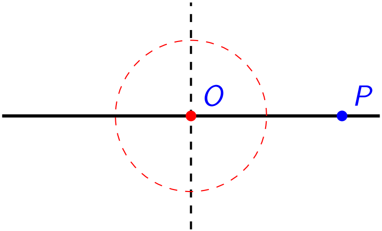
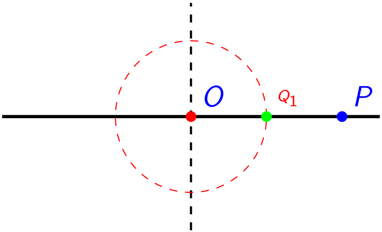
$x^4 - 1$ is the product of the distances from P to the vertices of a square inscribed in the unit circle!

Cotes : the genesis of the complex plane



What about $n = 3$?

Cotes : the genesis of the complex plane



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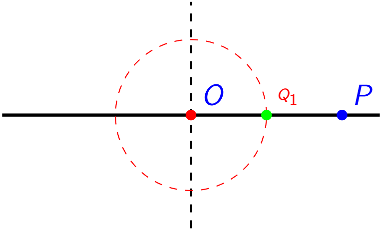

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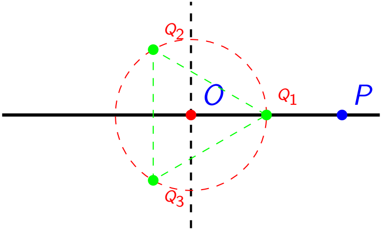

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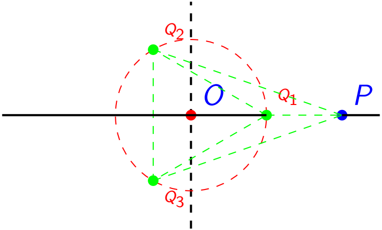

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Cotes : the genesis of the complex plane

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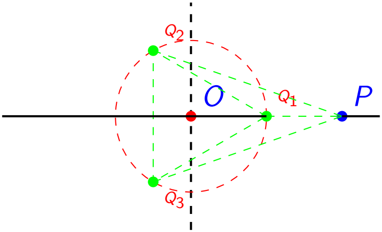

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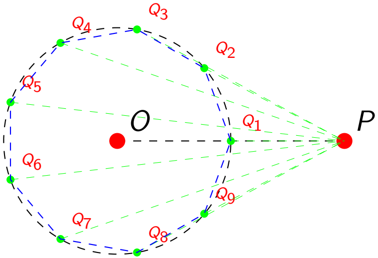
$$= \overline{PQ_1} \cdot \overline{PQ_2} \cdot \overline{PQ_3}$$

$x^3 - 1$ is the product the distances of the point P from the three vertices of an equilateral triangle inscibed in the unit circle.

Cotes : the genesis of the complex plane

If $Q_1 Q_2 \ldots Q_n$ is a regular n-gon inscribed in a circle of unit radius centered at O, and P is the point on $\overrightarrow{OQ_1}$ at distance x from O, then

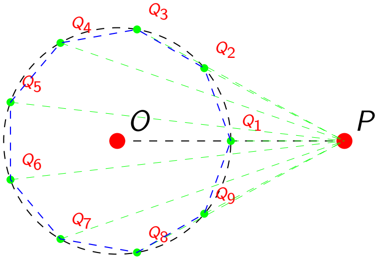
$$x^n - 1 = \overline{PQ_1} \cdot \overline{PQ_1} \cdot \ldots \cdot \overline{PQ_n}$$



Cotes : the genesis of the complex plane

If $Q_1 Q_2 \ldots Q_n$ is a regular n-gon inscribed in a circle of unit radius centered at O, and P is the point on $\overrightarrow{OQ_1}$ at distance x from O, then

$$x^n - 1 = \overline{PQ_1} \cdot \overline{PQ_1} \cdot \ldots \cdot \overline{PQ_n}$$



The roots of $x^n - 1 = 0$ are located at $Q_1, Q_2, \ldots, Q_n$!

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- He discovered the "mysterious" formula :

$$e^{\mathrm{i}\pi} = -1$$

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- The algebra of complex numbers!
- Hamilton went on to introduce ordered quadruples - *quaternions!*

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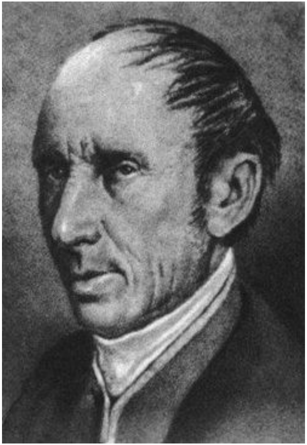
Analysis with complex numbers: Cauchy



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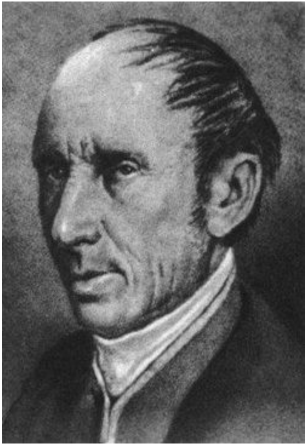

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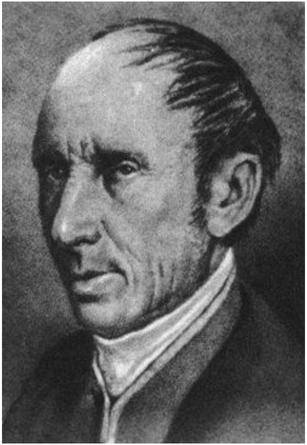

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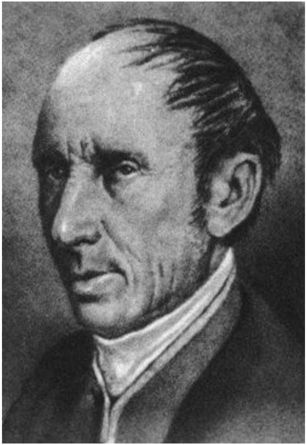

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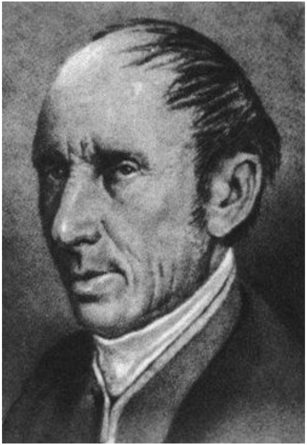

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- *"We completely repudiate the symbol $\sqrt{-1}$, abandoning it without regret because we do not know what this alleged symbolism signifies nor what meaning to give to it."*

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- Introduced the notion of Riemann surfaces.

The shortest path between two truths in the real domain passes through the complex domain.

Hadamard

Futher reading

- The MacTutor History of Mathematics archive : www-history.mcs.st-andrews.ac.uk
- Mathew Howell's list of internet history resources : math.fullerton.edu/mathews/c2003/HistoryComplexBib/Links/HistoryComplexBib_lnk_1.html
- A short history of complex numbers www.math.uri.edu/~merino/spring06/mth562/ShortHistoryComplexNumbers2006.pdf
- Cut-The-Knot : www.cut-the-knot.org/arithmetic/algebra/HistoricalRemarks.shtml