Harmonic functions

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MA211, Lecture 9

Harmonic function

If $\phi : \mathbb{R}^2 \to \mathbb{R}$ and the derivativs $\phi_x, \phi_y, \phi_{xx}, \phi_{yy}$ are all continuous and if $\phi(x, y)$ satisfies Laplace's equation

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Theorem

Let $f(z) = u(x, y) + \mathrm{i}v(x, y)$ be a holomorphic function in the domain $D \subset \mathbb{C}$. If all second order partial derivatives of u and v are continuous, then both u and v are harmonic functions in D.

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Let $f(z) = u(x, y) + \imath v(x, y)$ be a holomorphic function in the domain $D \subset \mathbb{C}$. If all second order partial derivatives of u and v are continuous, then both u and v are harmonic functions in D.

As we will see later, if $f(z)$ is holomorphic, then all partial derivatives of u and v are continuous - so that this holds for all holomorphic functions.

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So that u is harmonic - the proof for v is similar. □

Constructing harmonic functions

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$$\begin{aligned} u(x, y) &= x^3 - 3xy^2 \\ v(x, y) &= 3x^2y - y^3 \end{aligned}$$

Harmonic conjugates

If we are given a function $u(x, y)$ which is harmonic in $D \subset \mathbb{C}$ and if we can find a function $v(x, y)$ such that the partial derivatives of u and v satisfy the Cauchy-Riemann conditions everywhere in D then $v(x, y)$ is called the **harmonic conjugate** of $u(x, y)$.

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If such a $v(x, y)$ can be found, then

$$f(z) = u(x, y) + \imath v(x, y)$$

is holomorphic in D.

An example

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It is easy to see that this implies that the harmonic conjugate to u is

$$v(x,y) = 2xy + c$$

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- where all terms with x should cancel.
- Solving this gives us v.

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Thus u is harmonic

$$u_{xx} + u_{yy} = 0$$

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$$v_x(x,y) \;=\; -u_y(x,y) = -3xy^2 + x^3$$

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Thus

$$v_y(x,y) = u_x(x,y) \implies$$

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$$C(y) = \frac{1}{4}y^4 + c$$

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Thus

$$v(x,y) = \frac{1}{4}x^4 + \frac{1}{4}y^4 - \frac{3}{2}x^2y^2 + ic$$

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We can, of course, directly verify that the Laplace equation is obeyed by uv (▸ ?).

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- $f(z) = u + \imath v$ is holomorphic.
- So is $(f(z))^2$.
- Thus $uv = \frac{1}{2}\Im\left((f(z))^2\right)$ is harmonic.

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Complex analysis helps in all these physical applications and more ...

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Thus

$$\begin{aligned} w_{xx} + w_{yy} &= (u_{xx} + u_{yy}) v + 2 u_x v_x + 2 u_y v_y \\ &\quad + u (v_{xx} + u_{yy}) \end{aligned}$$

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Thus

$$\begin{aligned} w_{xx} + w_{yy} &= (u_{xx} + u_{yy}) v + 2u_x v_x + 2u_y v_y \\ &\quad + u (v_{xx} + u_{yy}) \\ &= 0 \cdot v + 2u_x (-u_y) + 2u_y u_x + u \cdot 0 \\ &= 0 \end{aligned}$$