Ananda Dasgupta

MA211, Lecture 9

If $\phi: \mathbb{R}^2 \to \mathbb{R}$ and the derivative $\phi_x, \phi_y, \phi_{xx}, \phi_{yy}$ are all continuous and if $\phi(x, y)$ satisfies Laplace's equation

$$\phi_{xx}(x,y) + \phi_{yy}(x,y) = 0$$

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As we will see later, if f(z) is holomorphic, then all partial derivatives of u and v are continuous - so that this holds for all holomorphic functions.

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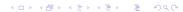
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So that u is harmonic - the proof for v is similar.



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Harmonic conjugates

If we are given a function u(x,y) which is harmonic in $D \subset \mathbb{C}$ and if we can find a function v(x,y) such that the partial derivatives of u and v satisfy the Cauchy-Riemann conditions everywhere in D then v(x,y) is called the **harmonic conjugate** of u(x,y).

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It is easy to see that this implies that the harmonic conjugate to u is

$$v(x,y) = 2xy + c$$



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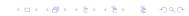
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- where all terms with x should cancel.
- ▶ Solving this gives us v.



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Thus *u* is harmonic

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$$C(y) = \frac{1}{4}y^4 + c$$

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$$v(x,y) = \frac{1}{4}x^4 + \frac{1}{4}y^4 - \frac{3}{2}x^2y^2 + ic$$



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Proof.

We can, of course, directly verify that the Laplace equation is obeyed by uv (\bigcirc).

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An elegant proof:

 \triangleright Since u and v are harmonic conjugates,

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- ► Thus $uv = \frac{1}{2}\Im\left(\left(f(z)\right)^2\right)$ is harmonic.

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Complex analysis helps in all these physical applications and more ...



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