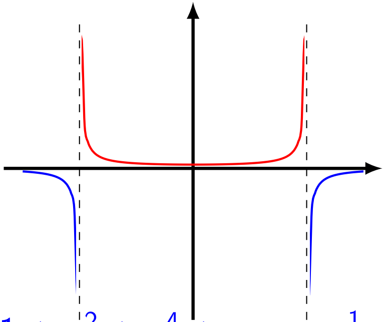
Complex Sequences

Ananda Dasgupta

MA211, Lecture 11

The mystery of real power series



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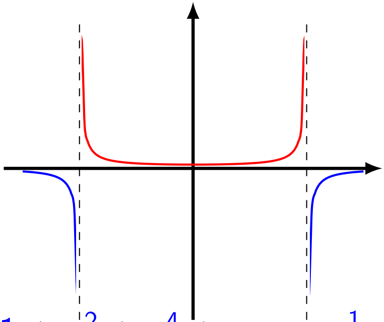

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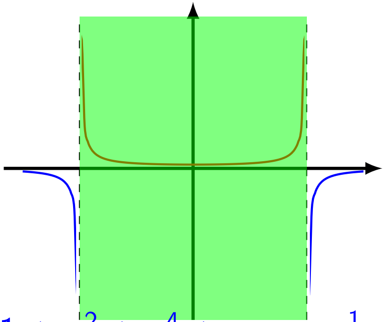

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The series can not continue to converge beyond the "walls of divergence"!

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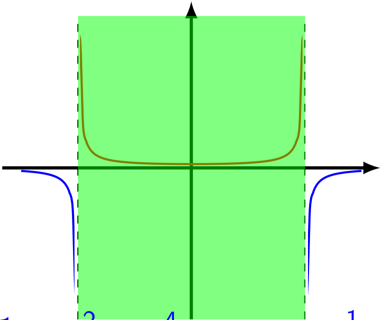

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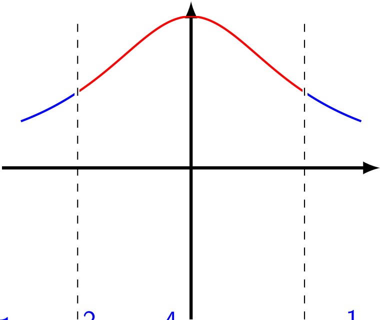

$1 - x^2 + x^4 + \ldots = \frac{1}{1+x^2}$

This series also converges only for $-1 < x < 1$.

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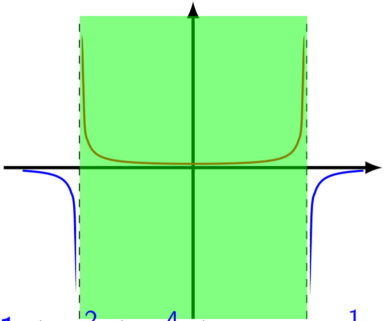

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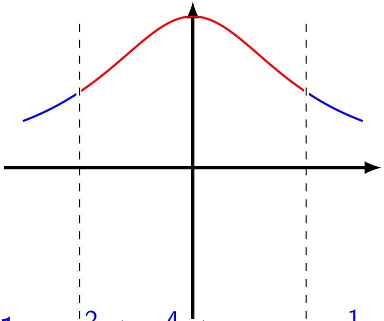

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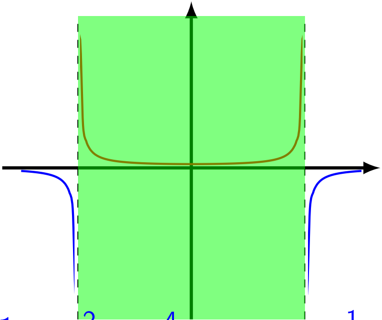

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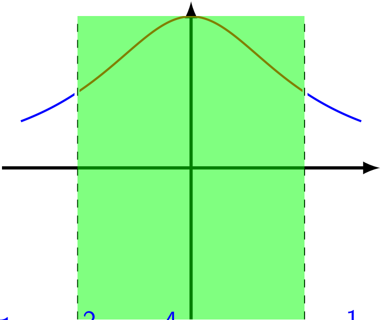

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This series also converges only for $-1 < x < 1$.
However, $\frac{1}{1+x^2}$ converges for all real x!
Why, then, does the series diverge for $|x| > 1$?

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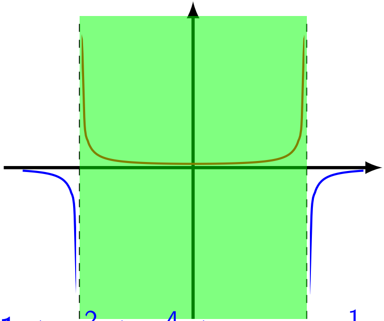

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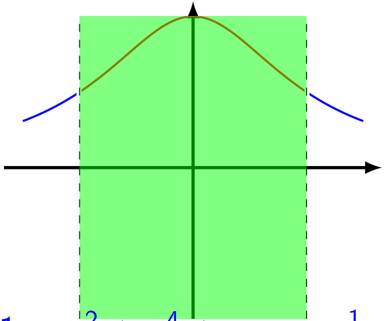

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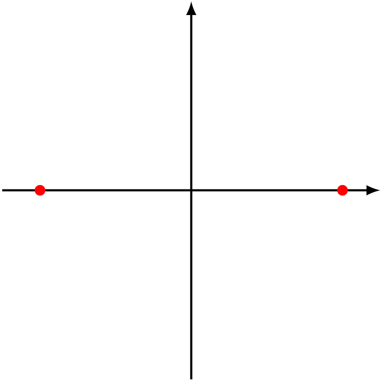

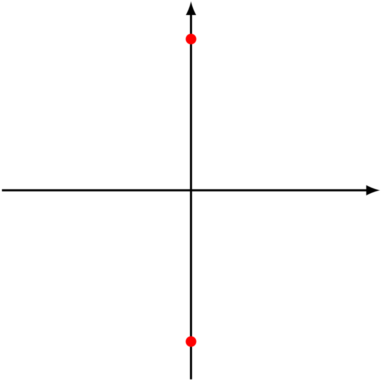

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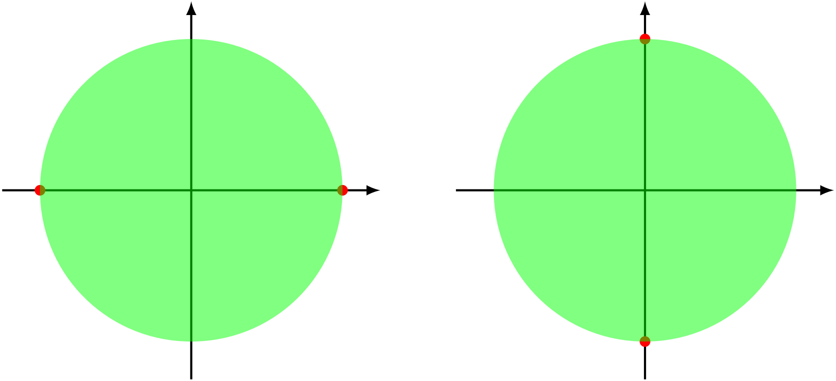

The mystery is easily resolved in the complex plane!
Both $\frac{1}{1 \pm z^2}$ have singularities.
The power series only converges up to the distance to the nearest singularity!

One more example

- Consider the series
$$\frac{1}{1-x} = 1 + x + x^2 + \dots$$

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- It converges for $|x| < 1$.
- Its complex counterpart $1 + z + z^2 + \dots$ converge within the unit circle $|z| < 1$.

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$$\begin{aligned}\frac{1}{1-x} &= 2\frac{1}{1-2\left(x-\frac{1}{2}\right)} \\ &= 2+2^2\left(x-\frac{1}{2}\right)+2^3\left(x-\frac{1}{2}\right)^2+\ldots\end{aligned}$$

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- This converges only for $\left|2\left(x-\frac{1}{2}\right)\right| < 1$.
- *i.e.* for $x \in (0,1)$
- Its complex cousin $2+2^2\left(z-\frac{1}{2}\right)+2^3\left(z-\frac{1}{2}\right)^2+\ldots$ converges everywhere in the disk $\left|z-\frac{1}{2}\right| < \frac{1}{2}$.

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- *only within the intersection of the two disks of convergence!*
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- Here both sides actually match the function $\frac{1}{1-z}$.

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- Can you identify the region where this series will converge?

In order to understand this better we must take a closer look at sequences and series of complex numbers.

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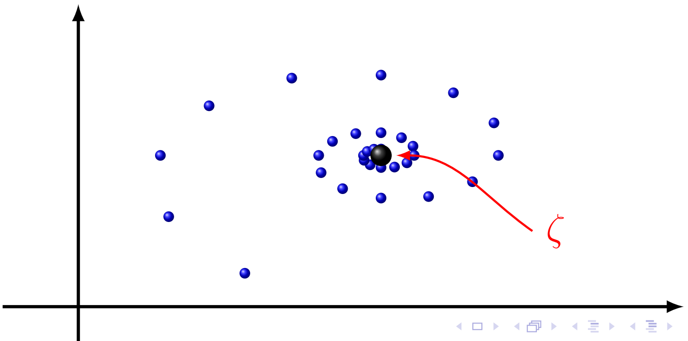
- This is often abbreviated to (z_n) or even as just z_n.

Limit of a sequence

A sequence z_n is said to converge to a limit ζ if $\forall \epsilon > 0$, $\exists N(\epsilon) \in \mathbb{N}$:

$$n \in \mathbb{N} > N(\epsilon) \implies |z_n - \zeta| < \epsilon$$



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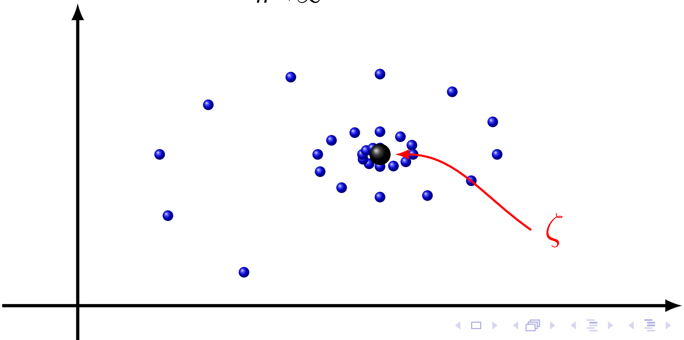
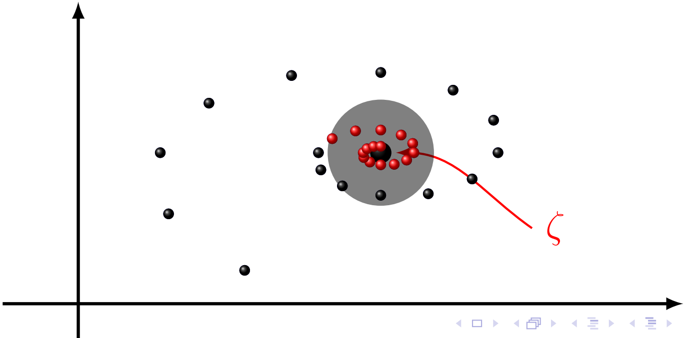
We write

$$\lim_{n\to\infty} z_n = \zeta$$



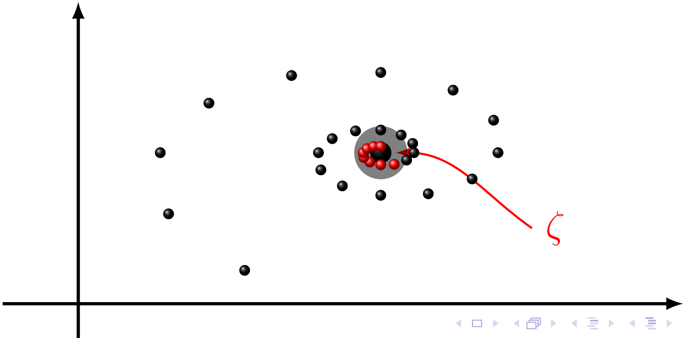
Limit of a sequence

No matter how finicky we are, we will be able to keep the **tail** *of the sequence confined to an* ϵ*-disk centered around* ζ *by throwing away a sufficiently large number of terms from the* **head***!*



Limit of a sequence

Choose a smaller ϵ and you may haved to throw away a larger number of terms from the head!



Connection with real sequences

Theorem

Let $z_n = x_n + \mathrm{i} y_n$ *and* $\zeta = u + \mathrm{i} v$ *then*

$$\lim_{n \to \infty} z_n = \zeta$$

iff $\lim_{n \to \infty} x_n = u$ *and* $\lim_{n \to \infty} y_n = v$.

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This means that we can study limits of complex sequences using our knowledge of real analysis.

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Let us assume that $u \neq v$. Then $D = |u - v| > 0$.

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$$D = |u - v| \;=\; |(z_n - v) - (z_n - u)|$$

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Which is a contradiction! $\square$

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$$D = |u - v| = |(z_n - v) - (z_n - u)| \leq |z_n - v| + |z_n - u| < \frac{D}{2} + \frac{D}{2} = D$$

Which is a contradiction! $\square$

It is not possible to have the tail arbitrarily close to two distinct points!

$\lim z \implies \lim x, \lim y$

$\lim_{n\to\infty} z_n = \zeta$ implies that $\forall \epsilon > 0$, $\exists N(\epsilon) \in \mathbb{N}$ such that

$$n > N(\epsilon) \implies |z_n - \zeta| < \epsilon.$$

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Then $n > N(\epsilon) \implies$

$$|x_n - u| \;=\; |\Re\,(z_n - \zeta)|$$

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Then $n > N(\epsilon) \implies$

$$\begin{aligned} |x_n - u| &= |\Re(z_n - \zeta)| \\ &\leq |z_n - \zeta| \end{aligned}$$

$\lim z \implies \lim x, \lim y$

$\lim_{n\to\infty} z_n = \zeta$ implies that $\forall \epsilon > 0$, $\exists N(\epsilon) \in \mathbb{N}$ such that

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Thus $\lim_{n\to\infty} x_n = u$.
The proof for v is similar.

$\lim x, \lim y \implies \lim z$

$$\lim_{n\to\infty} x_n = u, \ \lim_{n\to\infty} y_n = v \implies \exists N_1(\epsilon), N_2(\epsilon) \in \mathbb{N}$$

such that

$\lim x, \lim y \implies \lim z$

$$\lim_{n\to\infty} x_n = u, \ \lim_{n\to\infty} y_n = v \implies \exists N_1(\epsilon), N_2(\epsilon) \in \mathbb{N}$$

such that

$$n > N_1(\epsilon) \implies |x_n - u| < \frac{\epsilon}{2}$$

$\lim x, \lim y \implies \lim z$

$$\lim_{n\to\infty} x_n = u, \ \lim_{n\to\infty} y_n = v \implies \exists N_1(\epsilon), N_2(\epsilon) \in \mathbb{N}$$

such that

$$n > N_1(\epsilon) \implies |x_n - u| < \frac{\epsilon}{2}$$

$$n > N_2(\epsilon) \implies |y_n - v| < \frac{\epsilon}{2}$$

$\lim x, \lim y \implies \lim z$

$\lim_{n\to\infty} x_n = u, \ \lim_{n\to\infty} y_n = v \implies \exists N_1(\epsilon), N_2(\epsilon) \in \mathbb{N}$ such that

$$n > N_1(\epsilon) \implies |x_n - u| < \frac{\epsilon}{2}$$

$$n > N_2(\epsilon) \implies |y_n - v| < \frac{\epsilon}{2}$$

Choose $N(\epsilon) = \max\{N_1(\epsilon), N_2(\epsilon)\}$.

$\lim x, \lim y \implies \lim z$

Then $n > N(\epsilon) \implies$

$$|z_n - \zeta| = |x_n + \mathfrak{i} v_n - (u + \mathfrak{i} v)|$$

$\lim x, \lim y \implies \lim z$

Then $n > N(\epsilon) \implies$

$$\begin{aligned} |z_n - \zeta| &= |x_n + \mathrm{i} v_n - (u + \mathrm{i} v)| \\ &= |(x_n - u) + \mathrm{i}(y_n - v)| \end{aligned}$$

$\lim x, \lim y \implies \lim z$

Then $n > N(\epsilon) \implies$

$$\begin{aligned} |z_n - \zeta| &= |x_n + \mathrm{i} v_n - (u + \mathrm{i} v)| \\ &= |(x_n - u) + \mathrm{i}\,(y_n - v)| \\ &\leq |x_n - u| + |\mathrm{i}\,(y_n - v)| \end{aligned}$$

$\lim x, \lim y \implies \lim z$

Then $n > N(\epsilon) \implies$

$$\begin{aligned} |z_n - \zeta| &= |x_n + \mathfrak{i} v_n - (u + \mathfrak{i} v)| \\ &= |(x_n - u) + \mathfrak{i}\,(y_n - v)| \\ &\leq |x_n - u| + |\mathfrak{i}\,(y_n - v)| \\ &= |x_n - u| + |\mathfrak{i}|\,|y_n - v| \end{aligned}$$

$\lim x, \lim y \implies \lim z$

Then $n > N(\epsilon) \implies$

$$\begin{aligned}
|z_n - \zeta| &= |x_n + \mathrm{i} v_n - (u + \mathrm{i} v)| \\
&= |(x_n - u) + \mathrm{i}\,(y_n - v)| \\
&\leq |x_n - u| + |\mathrm{i}\,(y_n - v)| \\
&= |x_n - u| + |\mathrm{i}|\,|y_n - v| \\
&= |x_n - u| + |y_n - v|
\end{aligned}$$

$\lim x, \lim y \implies \lim z$

Then $n > N(\epsilon) \implies$

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|z_n - \zeta| &= |x_n + \mathrm{i}v_n - (u + \mathrm{i}v)| \\
&= |(x_n - u) + \mathrm{i}\,(y_n - v)| \\
&\leq |x_n - u| + |\mathrm{i}\,(y_n - v)| \\
&= |x_n - u| + |\mathrm{i}|\,|y_n - v| \\
&= |x_n - u| + |y_n - v| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2}
\end{aligned}$$

$\lim x, \lim y \implies \lim z$

Then $n > N(\epsilon) \implies$

$$\begin{aligned} |z_n - \zeta| &= |x_n + \mathrm{i} v_n - (u + \mathrm{i} v)| \\ &= |(x_n - u) + \mathrm{i}\,(y_n - v)| \\ &\leq |x_n - u| + |\mathrm{i}\,(y_n - v)| \\ &= |x_n - u| + |\mathrm{i}|\,|y_n - v| \\ &= |x_n - u| + |y_n - v| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

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