Complex Sequences contd.

Ananda Dasgupta

MA211, Lecture 12

Bounded sequences

A sequence (z_n) is called **bounded** if $\exists M \in \mathbb{R}^+$ such that $|z_n| \leq M \ \forall n$.

Bounded sequences

A sequence (z_n) is called **bounded** if $\exists M \in \mathbb{R}^+$ such that $|z_n| \leq M \ \forall n$.

Theorem

Every convergent sequence is bounded.

Bounded sequences

A sequence (z_n) is called **bounded** if $\exists M \in \mathbb{R}^+$ such that $|z_n| \leq M \; \forall n$.

Theorem

Every convergent sequence is bounded.

The converse, however, is not true!

Bounded sequences

A sequence (z_n) is called **bounded** if $\exists M \in \mathbb{R}^+$ such that $|z_n| \leq M \; \forall n$.

Theorem

Every convergent sequence is bounded.

The converse, however, is not true!

The sequence $z_n = e^{\mathrm{i}n\theta}$ is bounded

Bounded sequences

A sequence (z_n) is called **bounded** if $\exists M \in \mathbb{R}^+$ such that $|z_n| \leq M \; \forall n$.

Theorem

Every convergent sequence is bounded.

The converse, however, is not true!

The sequence $z_n = e^{in\theta}$ is bounded - but not convergent.

Bounded sequences

Proof.

Let $\lim_{n\to\infty} z_n = \zeta$. Then, $\exists N \in \mathbb{N}$:

$$n > N \implies |z_n - \zeta| < 1$$

Bounded sequences

Proof.

Let $\lim_{n\to\infty} z_n = \zeta$. Then, $\exists N \in \mathbb{N}$:

$$n > N \implies |z_n - \zeta| < 1$$

Then, for $n > N$, $|z_n| = |z_n - \zeta + \zeta|$

Bounded sequences

Proof.

Let $\lim_{n\to\infty} z_n = \zeta$. Then, $\exists N \in \mathbb{N}$:

$$n > N \implies |z_n - \zeta| < 1$$

Then, for $n > N$, $|z_n| = |z_n - \zeta + \zeta| < |\zeta| + 1$

Bounded sequences

Proof.

Let $\lim_{n\to\infty} z_n = \zeta$. Then, $\exists N \in \mathbb{N}$:

$$n > N \implies |z_n - \zeta| < 1$$

Then, for $n > N$, $|z_n| = |z_n - \zeta + \zeta| < |\zeta| + 1$

Then

$$M = max\left\{|z_1|, |z_2|, \ldots, |z_N|, 1 + |\zeta|\right\}$$

satisfies the definition of boundedness. □

Example

$$z_n = (1 + \mathfrak{i})^n$$

Example

$z_n = (1 + \mathfrak{i})^n$

This sequence is not bounded :

$$|z_n| = 2^{\frac{n}{2}}$$

Example

$z_n = (1 + \mathfrak{i})^n$
This sequence is not bounded :

$$|z_n| = 2^{\frac{n}{2}}$$

Thus z_n is not convergent!

Cauchy sequences

Theorem

If a sequence z_n *converges to a limit* ζ*, then* $\forall \epsilon > 0$, $\exists N(\epsilon) \in \mathbb{N}$ *such that* $n, m > N(\epsilon) \implies$

$$|z_n - z_m| < \epsilon.$$

Cauchy sequences

Proof.

Since $\lim_{n\to\infty} z_n = \zeta$, $\forall\epsilon > 0, \exists N(\epsilon) \in \mathbb{N}$ such that $n, m > N(\epsilon) \implies$

$$|z_n - \zeta|\,,\ |z_m - \zeta| < \frac{\epsilon}{2}$$

Cauchy sequences

Proof.

Since $\lim_{n\to\infty} z_n = \zeta$, $\forall\epsilon > 0, \exists N(\epsilon) \in \mathbb{N}$ such that $n, m > N(\epsilon) \implies$

$$|z_n - \zeta|\,,\ |z_m - \zeta| < \frac{\epsilon}{2}$$

Then

$$|z_n - z_m| \;=\; |(z_n - \zeta) - (z_m - \zeta)|$$

Cauchy sequences

Proof.

Since $\lim_{n\to\infty} z_n = \zeta$, $\forall\epsilon > 0, \exists N(\epsilon) \in \mathbb{N}$ such that $n, m > N(\epsilon) \implies$

$$|z_n - \zeta|\,,\ |z_m - \zeta| < \frac{\epsilon}{2}$$

Then

$$\begin{aligned} |z_n - z_m| &= |(z_n - \zeta) - (z_m - \zeta)| \\ &\leq |z_n - \zeta| + |z_m - \zeta| \end{aligned}$$

Cauchy sequences

Proof.

Since $\lim_{n\to\infty} z_n = \zeta$, $\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}$ such that $n, m > N(\epsilon) \implies$

$$|z_n - \zeta|, \; |z_m - \zeta| < \frac{\epsilon}{2}$$

Then

$$\begin{aligned} |z_n - z_m| &= |(z_n - \zeta) - (z_m - \zeta)| \\ &\leq |z_n - \zeta| + |z_m - \zeta| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \end{aligned}$$

Cauchy sequences

Proof.

Since $\lim_{n\to\infty} z_n = \zeta$, $\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}$ such that $n, m > N(\epsilon) \implies$

$$|z_n - \zeta|\,,\ |z_m - \zeta| < \frac{\epsilon}{2}$$

Then

$$\begin{aligned} |z_n - z_m| &= |(z_n - \zeta) - (z_m - \zeta)| \\ &\leq |z_n - \zeta| + |z_m - \zeta| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

$\square$

Cauchy sequences

A sequence z_n is called a **Cauchy sequence** if $\forall \epsilon > 0$, $\exists N(\epsilon) \in \mathbb{N}$ such that $n, m > N(\epsilon) \implies$

$$|z_n - z_m| < \epsilon.$$

Cauchy sequences

A sequence z_n is called a **Cauchy sequence** if $\forall \epsilon > 0$, $\exists N(\epsilon) \in \mathbb{N}$ such that $n, m > N(\epsilon) \implies$

$$|z_n - z_m| < \epsilon.$$

A convergent sequence is always a Cauchy sequence.

Cauchy sequences

A sequence z_n is called a **Cauchy sequence** if $\forall \epsilon > 0$, $\exists N(\epsilon) \in \mathbb{N}$ such that $n, m > N(\epsilon) \implies$

$$|z_n - z_m| < \epsilon.$$

A convergent sequence is always a Cauchy sequence.
Is the converse true?

Cauchy sequences

A sequence z_n is called a **Cauchy sequence** if $\forall \epsilon > 0$, $\exists N(\epsilon) \in \mathbb{N}$ such that $n, m > N(\epsilon) \implies$

$$|z_n - z_m| < \epsilon.$$

A convergent sequence is always a Cauchy sequence.
Is the converse true? Yes!

Cauchy sequences

A sequence z_n is called a **Cauchy sequence** if $\forall \epsilon > 0$, $\exists N(\epsilon) \in \mathbb{N}$ such that $n, m > N(\epsilon) \implies$

$$|z_n - z_m| < \epsilon.$$

A convergent sequence is always a Cauchy sequence. Is the converse true? Yes! $\mathbb{C}$ is **Cauchy complete**!

Cauchy sequences

A sequence z_n is called a **Cauchy sequence** if $\forall \epsilon > 0$, $\exists N(\epsilon) \in \mathbb{N}$ such that $n, m > N(\epsilon) \implies$

$$|z_n - z_m| < \epsilon.$$

A convergent sequence is always a Cauchy sequence.

Is the converse true? Yes! $\mathbb{C}$ is **Cauchy complete**!

Every Cauchy sequence in $\mathbb{C}$ converges to a limit.

▸ Proof

Cauchy sequences

A sequence z_n is called a **Cauchy sequence** if $\forall \epsilon > 0$, $\exists N(\epsilon) \in \mathbb{N}$ such that $n, m > N(\epsilon) \implies$

$$|z_n - z_m| < \epsilon.$$

A convergent sequence is always a Cauchy sequence.
Is the converse true? Yes! $\mathbb{C}$ is **Cauchy complete**!
Every Cauchy sequence in $\mathbb{C}$ converges to a limit.

▸ Proof

Contrast : *On* $\mathbb{Q}$ *the sequence*

$$1, 1.4, 1.41, 1.414, 1.4142, \ldots$$

is a Cauchy sequence, but does not converge!

Some properties of sequences

Let z_n and w_n be two convergent sequences converging to the limits ζ and ω, respectively. Then

Some properties of sequences

Let z_n and w_n be two convergent sequences converging to the limits ζ and ω, respectively. Then

- The sequence cz_n $(c \in \mathbb{C})$ converges to $c\zeta$ ▸ Proof.

Some properties of sequences

Let z_n and w_n be two convergent sequences converging to the limits ζ and ω, respectively. Then

- The sequence cz_n $(c \in \mathbb{C})$ converges to $c\zeta$ Proof.
- The sequence $z_n + w_n$ converges to $\zeta + \omega$.

Some properties of sequences

Let z_n and w_n be two convergent sequences converging to the limits ζ and ω, respectively. Then

- The sequence cz_n $(c \in \mathbb{C})$ converges to $c\zeta$ ▸ Proof.
- The sequence $z_n + w_n$ converges to $\zeta + \omega$.
- The sequence $z_n w_n$ converges to $\zeta\omega$ ▸ Proof.

Some properties of sequences

Let z_n and w_n be two convergent sequences converging to the limits ζ and ω, respectively. Then

- The sequence cz_n $(c \in \mathbb{C})$ converges to $c\zeta$ Proof.
- The sequence $z_n + w_n$ converges to $\zeta + \omega$.
- The sequence $z_n w_n$ converges to $\zeta\omega$ Proof.
- The sequence $\frac{z_n}{w_n}$ is

Some properties of sequences

Let z_n and w_n be two convergent sequences converging to the limits ζ and ω, respectively. Then

- The sequence cz_n $(c \in \mathbb{C})$ converges to $c\zeta$ ▸ Proof.
- The sequence $z_n + w_n$ converges to $\zeta + \omega$.
- The sequence $z_n w_n$ converges to $\zeta\omega$ ▸ Proof.
- The sequence $\frac{z_n}{w_n}$ is
  - well defined if all $w_n \neq 0$

Some properties of sequences

Let z_n and w_n be two convergent sequences converging to the limits ζ and ω, respectively. Then

- The sequence cz_n $(c \in \mathbb{C})$ converges to $c\zeta$ ▸ Proof.
- The sequence $z_n + w_n$ converges to $\zeta + \omega$.
- The sequence $z_n w_n$ converges to $\zeta\omega$ ▸ Proof.
- The sequence $\frac{z_n}{w_n}$ is
  - well defined if all $w_n \neq 0$
  - converges to the limit $\frac{\zeta}{\omega}$ if $\omega \neq 0$

Some properties of sequences

Let z_n and w_n be two convergent sequences converging to the limits ζ and ω, respectively. Then

- The sequence cz_n $(c \in \mathbb{C})$ converges to $c\zeta$ ▸ Proof.
- The sequence $z_n + w_n$ converges to $\zeta + \omega$.
- The sequence $z_n w_n$ converges to $\zeta\omega$ ▸ Proof.
- The sequence $\frac{z_n}{w_n}$ is
  - well defined if all $w_n \neq 0$
  - converges to the limit $\frac{\zeta}{\omega}$ if $\omega \neq 0$
  - diverges if $\omega = 0$ ▸ Proof.

Proof that $\mathbb{C}$ is Cauchy complete

We will assume the result from real analysis that $\mathbb{R}$ is Cauchy complete.

Proof that $\mathbb{C}$ is Cauchy complete

We will assume the result from real analysis that $\mathbb{R}$ is Cauchy complete.

$z_n = x_n + \imath y_n$ being a Cauchy sequence, $\forall \epsilon > 0$, $\exists N(\epsilon) \in \mathbb{N}$ such that

$$n, m > N(\epsilon) \implies |z_n - z_m| < \epsilon$$

Proof that $\mathbb{C}$ is Cauchy complete

We will assume the result from real analysis that $\mathbb{R}$ is Cauchy complete.

$z_n = x_n + \imath y_n$ being a Cauchy sequence, $\forall \epsilon > 0$, $\exists N(\epsilon) \in \mathbb{N}$ such that

$$n, m > N(\epsilon) \implies |z_n - z_m| < \epsilon$$

But then,

$$|x_n - x_m| \leq |z_n - z_m|$$

Proof that $\mathbb{C}$ is Cauchy complete

We will assume the result from real analysis that $\mathbb{R}$ is Cauchy complete.

$z_n = x_n + \imath y_n$ being a Cauchy sequence, $\forall \epsilon > 0$, $\exists N(\epsilon) \in \mathbb{N}$ such that

$$n, m > N(\epsilon) \implies |z_n - z_m| < \epsilon$$

But then,

$$|x_n - x_m| \leq |z_n - z_m| < \epsilon$$

Proof that $\mathbb{C}$ is Cauchy complete

We will assume the result from real analysis that $\mathbb{R}$ is Cauchy complete.

$z_n = x_n + \imath y_n$ being a Cauchy sequence, $\forall \epsilon > 0$, $\exists N(\epsilon) \in \mathbb{N}$ such that

$$n, m > N(\epsilon) \implies |z_n - z_m| < \epsilon$$

But then,

$$|x_n - x_m| \leq |z_n - z_m| < \epsilon$$

Thus the real sequence x_n is a Cauchy sequence

Proof that $\mathbb{C}$ is Cauchy complete

We will assume the result from real analysis that $\mathbb{R}$ is Cauchy complete.

$z_n = x_n + \imath y_n$ being a Cauchy sequence, $\forall \epsilon > 0$, $\exists N(\epsilon) \in \mathbb{N}$ such that

$$n, m > N(\epsilon) \implies |z_n - z_m| < \epsilon$$

But then,

$$|x_n - x_m| \leq |z_n - z_m| < \epsilon$$

Thus the real sequence x_n is a Cauchy sequence and is thus convergent.

Proof that $\mathbb{C}$ is Cauchy complete

We will assume the result from real analysis that $\mathbb{R}$ is Cauchy complete.

$z_n = x_n + \imath y_n$ being a Cauchy sequence, $\forall \epsilon > 0$, $\exists N(\epsilon) \in \mathbb{N}$ such that

$$n, m > N(\epsilon) \implies |z_n - z_m| < \epsilon$$

But then,

$$|x_n - x_m| \leq |z_n - z_m| < \epsilon$$

Thus the real sequence x_n is a Cauchy sequence and is thus convergent.

The same argument works for y_n also!

Proof that $\mathbb{C}$ is Cauchy complete

We will assume the result from real analysis that $\mathbb{R}$ is Cauchy complete.

$z_n = x_n + \imath y_n$ being a Cauchy sequence, $\forall \epsilon > 0$, $\exists N(\epsilon) \in \mathbb{N}$ such that

$$n, m > N(\epsilon) \implies |z_n - z_m| < \epsilon$$

But then,

$$|x_n - x_m| \leq |z_n - z_m| < \epsilon$$

Thus the real sequence x_n is a Cauchy sequence and is thus convergent.

The same argument works for y_n also!

Hence z_n is convergent sequence.

$$\lim_{n\to\infty} cz_n = c\zeta$$

Proof.

Since $\lim_{n\to\infty} z_n = \zeta$

$\lim_{n \to \infty} cz_n = c\zeta$

Proof.

Since $\lim_{n \to \infty} z_n = \zeta$

$\forall \epsilon > 0$, $\exists N(\epsilon) \in \mathbb{N}$ such that $n > N(\epsilon) \implies$

$$|z_n - \zeta| < \frac{\epsilon}{|c|}$$

$\lim_{n \to \infty} c z_n = c\zeta$

Proof.

Since $\lim_{n \to \infty} z_n = \zeta$

$\forall \epsilon > 0$, $\exists N(\epsilon) \in \mathbb{N}$ such that $n > N(\epsilon) \implies$

$$|z_n - \zeta| < \frac{\epsilon}{|c|}$$

Then

$$|c z_n - c\zeta| = |c|\,|z_n - \zeta|$$

$\lim_{n\to\infty} cz_n = c\zeta$

Proof.

Since $\lim_{n\to\infty} z_n = \zeta$

$\forall \epsilon > 0$, $\exists N(\epsilon) \in \mathbb{N}$ such that $n > N(\epsilon) \implies$

$$|z_n - \zeta| < \frac{\epsilon}{|c|}$$

Then

$$\begin{aligned} |cz_n - c\zeta| &= |c|\,|z_n - \zeta| \\ &< |c|\,\frac{\epsilon}{|c|} \end{aligned}$$

$$\lim_{n\to\infty} cz_n = c\zeta$$

Proof.

Since $\lim_{n\to\infty} z_n = \zeta$

$\forall \epsilon > 0$, $\exists N(\epsilon) \in \mathbb{N}$ such that $n > N(\epsilon) \implies$

$$|z_n - \zeta| < \frac{\epsilon}{|c|}$$

Then

$$\begin{aligned} |cz_n - c\zeta| &= |c|\,|z_n - \zeta| \\ &< |c|\,\frac{\epsilon}{|c|} = \epsilon \end{aligned}$$

□

$$\lim_{n\to\infty} z_n w_n = \zeta\omega$$

Proof.

$$|z_n w_n - \zeta\omega| \;=\; |z_n w_n - \zeta w_n + \zeta w_n - \zeta\omega|$$

$\lim_{n \to \infty} z_n w_n = \zeta\omega$

Proof.

$$\begin{aligned} |z_n w_n - \zeta\omega| &= |z_n w_n - \zeta w_n + \zeta w_n - \zeta\omega| \\ &= |(z_n - \zeta)\, w_n + \zeta\,(w_n - \omega)| \end{aligned}$$

$\lim_{n\to\infty} z_n w_n = \zeta\omega$

Proof.

$$\begin{aligned} |z_n w_n - \zeta\omega| &= |z_n w_n - \zeta w_n + \zeta w_n - \zeta\omega| \\ &= |(z_n - \zeta)\, w_n + \zeta\,(w_n - \omega)| \\ &\leq |z_n - \zeta|\,|w_n| + |\zeta|\,|w_n - \omega| \end{aligned}$$

$\lim_{n\to\infty} z_n w_n = \zeta\omega$

Proof.

$$|z_n w_n - \zeta\omega| \leq |z_n - \zeta|\,|w_n| + |\zeta|\,|w_n - \omega|$$

Since (w_n) converges, it is bounded,

$$\lim_{n \to \infty} z_n w_n = \zeta \omega$$

Proof.

$$|z_n w_n - \zeta\omega| \leq |z_n - \zeta|\,|w_n| + |\zeta|\,|w_n - \omega|$$

Since (w_n) converges, it is bounded,

$$\exists M' \in \mathbb{R}^+, \ : \ |w_n| \leq M', \ \forall n$$

$$\lim_{n\to\infty} z_n w_n = \zeta\omega$$

Proof.

$$|z_n w_n - \zeta\omega| \leq |z_n - \zeta|\,|w_n| + |\zeta|\,|w_n - \omega|$$

Since (w_n) converges, it is bounded,

$$\exists M' \in \mathbb{R}^+, \ : \ |w_n| \leq M', \ \forall n$$

Choose $M = max\{M', |\zeta|\}$.

$\lim_{n\to\infty} z_n w_n = \zeta\omega$

Proof.

$$|z_n w_n - \zeta\omega| \leq |z_n - \zeta|\,|w_n| + |\zeta|\,|w_n - \omega|$$

Since (w_n) converges, it is bounded,

$$\exists M' \in \mathbb{R}^+, \ : \ |w_n| \leq M', \ \forall n$$

Choose $M = max\{M', |\zeta|\}$.
Now $\forall \epsilon > 0, \ \exists N_1(\epsilon), N_2(\epsilon) \in \mathbb{N}$ such that

$$\lim_{n\to\infty} z_n w_n = \zeta\omega$$

Proof.

$$|z_n w_n - \zeta\omega| \leq |z_n - \zeta|\,|w_n| + |\zeta|\,|w_n - \omega|$$

Since (w_n) converges, it is bounded,

$$\exists M' \in \mathbb{R}^+, \ : \ |w_n| \leq M', \ \forall n$$

Choose $M = max\{M', |\zeta|\}$.
Now $\forall \epsilon > 0, \ \exists N_1(\epsilon), N_2(\epsilon) \in \mathbb{N}$ such that

$$n > N_1(\epsilon) \implies |z_n - \zeta| < \frac{\epsilon}{2M}$$

$\lim_{n\to\infty} z_n w_n = \zeta\omega$

Proof.

$$|z_n w_n - \zeta\omega| \leq |z_n - \zeta|\,|w_n| + |\zeta|\,|w_n - \omega|$$

Since (w_n) converges, it is bounded,

$$\exists M' \in \mathbb{R}^+, \ : \ |w_n| \leq M', \ \forall n$$

Choose $M = max\{M', |\zeta|\}$.
Now $\forall \epsilon > 0, \ \exists N_1(\epsilon), N_2(\epsilon) \in \mathbb{N}$ such that

$$n > N_1(\epsilon) \implies |z_n - \zeta| < \frac{\epsilon}{2M}$$
$$n > N_2(\epsilon) \implies |w_n - \omega| < \frac{\epsilon}{2M}$$

$\lim_{n\to\infty} z_n w_n = \zeta\omega$

Proof.

$$|z_n w_n - \zeta\omega| \le |z_n - \zeta|\,|w_n| + |\zeta|\,|w_n - \omega|$$

Since (w_n) converges, it is bounded,

$$\exists M' \in \mathbb{R}^+, \; : \; |w_n| \le M', \; \forall n$$

Choose $M = max\{M', |\zeta|\}$.
Now $\forall \epsilon > 0, \; \exists N_1(\epsilon), N_2(\epsilon) \in \mathbb{N}$ such that

$$n > N_1(\epsilon) \implies |z_n - \zeta| < \frac{\epsilon}{2M}$$
$$n > N_2(\epsilon) \implies |w_n - \omega| < \frac{\epsilon}{2M}$$

$\lim_{n\to\infty} z_n w_n = \zeta\omega$

$$|z_n w_n - \zeta\omega| \leq |z_n - \zeta|\,|w_n| + |\zeta|\,|w_n - \omega|$$

Choose $N(\epsilon) = \max\{N_1(\epsilon), N_2(\epsilon)\}$.

$$\lim_{n\to\infty} z_n w_n = \zeta\omega$$

$$|z_n w_n - \zeta\omega| \leq |z_n - \zeta|\,|w_n| + |\zeta|\,|w_n - \omega|$$

Choose $N(\epsilon) = \max\{N_1(\epsilon), N_2(\epsilon)\}$.

Then $n > N(\epsilon) \implies$

$$\lim_{n\to\infty} z_n w_n = \zeta\omega$$

$$|z_n w_n - \zeta\omega| \leq |z_n - \zeta|\,|w_n| + |\zeta|\,|w_n - \omega|$$

Choose $N(\epsilon) = \max\{N_1(\epsilon), N_2(\epsilon)\}$.

Then $n > N(\epsilon) \implies$

$$|z_n w_n - \zeta\omega| \;<\; \frac{\epsilon}{2M}|w_n| + |\zeta|\frac{\epsilon}{2M}$$

$$\lim_{n\to\infty} z_n w_n = \zeta\omega$$

$$|z_n w_n - \zeta\omega| \le |z_n - \zeta|\,|w_n| + |\zeta|\,|w_n - \omega|$$

Choose $N(\epsilon) = \max\{N_1(\epsilon), N_2(\epsilon)\}$.
Then $n > N(\epsilon) \implies$

$$\begin{aligned} |z_n w_n - \zeta\omega| &< \frac{\epsilon}{2M}|w_n| + |\zeta|\frac{\epsilon}{2M} \\ &\le \frac{\epsilon}{2M}M + \frac{\epsilon}{2M}M \end{aligned}$$

$$\lim_{n\to\infty} z_n w_n = \zeta\omega$$

$$|z_n w_n - \zeta\omega| \le |z_n - \zeta|\,|w_n| + |\zeta|\,|w_n - \omega|$$

Choose $N(\epsilon) = \max\{N_1(\epsilon), N_2(\epsilon)\}$.
Then $n > N(\epsilon) \implies$

$$\begin{aligned} |z_n w_n - \zeta\omega| &< \frac{\epsilon}{2M}|w_n| + |\zeta|\frac{\epsilon}{2M} \\ &\le \frac{\epsilon}{2M}M + \frac{\epsilon}{2M}M = \epsilon \end{aligned}$$

□

Go Back!

$$\lim_{n\to\infty} \frac{z_n}{w_n} = \frac{\zeta}{\omega}$$

It suffices to prove that $\lim_{n\to\infty} \frac{1}{w_n} = \frac{1}{\omega}$, given that $\forall n,\ w_n \neq 0$ and $\omega \neq 0$.

$$\lim_{n\to\infty} \frac{z_n}{w_n} = \frac{\zeta}{\omega}$$

It suffices to prove that $\lim_{n\to\infty} \frac{1}{w_n} = \frac{1}{\omega}$, given that $\forall n, \; w_n \neq 0$ and $\omega \neq 0$.

Since $w_n \neq 0$ and $\omega = \lim_{n\to\infty} w_n \neq 0$,

$$\exists m > 0 \; : \; \forall n, |w_n| > m$$

$$\lim_{n\to\infty} \frac{z_n}{w_n} = \frac{\zeta}{\omega}$$

It suffices to prove that $\lim_{n\to\infty} \frac{1}{w_n} = \frac{1}{\omega}$, given that $\forall n,\ w_n \neq 0$ and $\omega \neq 0$.

Since $w_n \neq 0$ and $\omega = \lim_{n\to\infty} w_n \neq 0$,

$$\exists m > 0 \ : \ \forall n, |w_n| > m$$

Since $\lim_{n\to\infty} w_n = \omega, \forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}$:

$$n > N(\epsilon) \implies |w_n - \omega| < \epsilon m|\omega|$$

$$\lim_{n\to\infty} \frac{z_n}{w_n} = \frac{\zeta}{\omega}$$

It suffices to prove that $\lim_{n\to\infty} \frac{1}{w_n} = \frac{1}{\omega}$, given that $\forall n,\ w_n \neq 0$ and $\omega \neq 0$.

Since $w_n \neq 0$ and $\omega = \lim_{n\to\infty} w_n \neq 0$,

$$\exists m > 0 \ : \ \forall n, |w_n| > m$$

Since $\lim_{n\to\infty} w_n = \omega, \forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}$:

$$n > N(\epsilon) \implies |w_n - \omega| < \epsilon m|\omega|$$

Then

$$\left|\frac{1}{w_n} - \frac{1}{\omega}\right| = \left|\frac{\omega - w_n}{\omega w_n}\right|$$

$$\lim_{n\to\infty} \frac{z_n}{w_n} = \frac{\zeta}{\omega}$$

It suffices to prove that $\lim_{n\to\infty} \frac{1}{w_n} = \frac{1}{\omega}$, given that $\forall n,\ \ w_n \neq 0$ and $\omega \neq 0$.

Since $w_n \neq 0$ and $\omega = \lim_{n\to\infty} w_n \neq 0$,

$$\exists m > 0 \ : \ \forall n, |w_n| > m$$

Since $\lim_{n\to\infty} w_n = \omega, \forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}$:

$$n > N(\epsilon) \implies |w_n - \omega| < \epsilon m|\omega|$$

Then

$$\left|\frac{1}{w_n} - \frac{1}{\omega}\right| = \left|\frac{\omega - w_n}{\omega w_n}\right| = \frac{|w_n - \omega|}{|\omega|\,|w_n|}$$

$$\lim_{n\to\infty} \frac{z_n}{w_n} = \frac{\zeta}{\omega}$$

It suffices to prove that $\lim_{n\to\infty} \frac{1}{w_n} = \frac{1}{\omega}$, given that $\forall n,\ w_n \neq 0$ and $\omega \neq 0$.

Since $w_n \neq 0$ and $\omega = \lim_{n\to\infty} w_n \neq 0$,

$$\exists m > 0 \ : \ \forall n, |w_n| > m$$

Since $\lim_{n\to\infty} w_n = \omega, \forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}$:

$$n > N(\epsilon) \implies |w_n - \omega| < \epsilon m|\omega|$$

Then

$$\left|\frac{1}{w_n} - \frac{1}{\omega}\right| = \left|\frac{\omega - w_n}{\omega w_n}\right| = \frac{|w_n - \omega|}{|\omega|\,|w_n|} < \frac{\epsilon m||\omega|}{|\omega|\,|w_n|}$$

$$\lim_{n\to\infty} \frac{z_n}{w_n} = \frac{\zeta}{\omega}$$

It suffices to prove that $\lim_{n\to\infty} \frac{1}{w_n} = \frac{1}{\omega}$, given that $\forall n,\ w_n \neq 0$ and $\omega \neq 0$.

Since $w_n \neq 0$ and $\omega = \lim_{n\to\infty} w_n \neq 0$,

$$\exists m > 0 \ : \ \forall n, |w_n| > m$$

Since $\lim_{n\to\infty} w_n = \omega, \forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}$:

$$n > N(\epsilon) \implies |w_n - \omega| < \epsilon m|\omega|$$

Then

$$\left|\frac{1}{w_n} - \frac{1}{\omega}\right| = \left|\frac{\omega - w_n}{\omega w_n}\right| = \frac{|w_n - \omega|}{|\omega|\,|w_n|} < \frac{\epsilon m|\omega|}{|\omega|\,|w_n|} < \epsilon \qquad \square$$

$\lim_{n\to\infty} \frac{1}{w_n}$ when $\omega = \lim_{n\to\infty} w_n = 0$

Let us assume that the limit $\lim_{n\to\infty} \frac{1}{w_n}$ exists and is equal to W.

Let us assume that the limit $\lim_{n\to\infty} \frac{1}{w_n}$ exists and is equal to W.
Then

1

$$\lim_{n\to\infty} \frac{1}{w_n} \text{ when } \omega = \lim_{n\to\infty} w_n = 0$$

Let us assume that the limit $\lim_{n\to\infty} \frac{1}{w_n}$ exists and is equal to W.

Then

$$1 = \lim_{n\to\infty} w_n \frac{1}{w_n}$$

$$\lim_{n\to\infty} \frac{1}{w_n} \text{ when } \omega = \lim_{n\to\infty} w_n = 0$$

Let us assume that the limit $\lim_{n\to\infty} \frac{1}{w_n}$ exists and is equal to W.

Then

$$1 = \lim_{n\to\infty} w_n \frac{1}{w_n} = \lim_{n\to\infty} w_n \lim_{n\to\infty} \frac{1}{w_n}$$

$$\lim_{n\to\infty} \frac{1}{w_n} \text{ when } \omega = \lim_{n\to\infty} w_n = 0$$

Let us assume that the limit $\lim_{n\to\infty} \frac{1}{w_n}$ exists and is equal to W.

Then

$$1 = \lim_{n\to\infty} w_n \frac{1}{w_n} = \lim_{n\to\infty} w_n \lim_{n\to\infty} \frac{1}{w_n} = \omega \times W$$

$\lim_{n\to\infty} \frac{1}{w_n}$ when $\omega = \lim_{n\to\infty} w_n = 0$

Let us assume that the limit $\lim_{n\to\infty} \frac{1}{w_n}$ exists and is equal to W.
Then

$$1 = \lim_{n\to\infty} w_n \frac{1}{w_n} = \lim_{n\to\infty} w_n \lim_{n\to\infty} \frac{1}{w_n} = \omega \times W = 0 \times W$$

$$\lim_{n\to\infty} \frac{1}{w_n} \text{ when } \omega = \lim_{n\to\infty} w_n = 0$$

Let us assume that the limit $\lim_{n\to\infty} \frac{1}{w_n}$ exists and is equal to W.
Then

$$1 = \lim_{n\to\infty} w_n \frac{1}{w_n} = \lim_{n\to\infty} w_n \lim_{n\to\infty} \frac{1}{w_n} = \omega \times W = 0 \times W$$

Which is impossible!

$$\lim_{n\to\infty} \frac{1}{w_n} \text{ when } \omega = \lim_{n\to\infty} w_n = 0$$

Let us assume that the limit $\lim_{n\to\infty} \frac{1}{w_n}$ exists and is equal to W.

Then

$$1 = \lim_{n\to\infty} w_n \frac{1}{w_n} = \lim_{n\to\infty} w_n \lim_{n\to\infty} \frac{1}{w_n} = \omega \times W = 0 \times W$$

Which is impossible!

Thus $\lim_{n\to\infty} \frac{1}{w_n}$ does not exist.

Go Back!