Sequences continued

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MA211, Lecture 13

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- A real sequence converges iff it is Cauchy.

Diverging to $\pm\infty$

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For sequences that converge or that diverge to $\pm\infty$ *the limit exists on the extended real line*

$$\mathbb{R}^* = \mathbb{R} \cup \{-\infty, +\infty\}.$$

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  - Here $N(M) = \text{int}(M) + 1$.
- The sequence $a_n = -\alpha n^2,\ \alpha > 0$ diverges to $-\infty$.
  - Here $N(M) = \text{int}\left(\sqrt{M/\alpha}\right) + 1$.

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So for $N(M)$ in our definition we can choose
$\max\{N'(M), 1\}$
Thus

$$\lim_{n\to\infty} a_n = +\infty$$

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Note that for these properties - the fact that $\mathbb{R}$ is ordered is crucial - in particular, they don't make sense for complex sequences!

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$\square$

Subsequences

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Theorem

Let $a = (a_n)_{n\in\mathbb{N}}$ be a convergent sequence. Then every subsequence $a = (a_{n_k})$ of $a = (a_n)$ is a convergent sequence and (?)

$$\lim a_{n_k} = \lim a_n$$

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Bolzano-Weirstrass theorem : Every bounded sequence of real numbers has at least one cluster point.

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Both the sequences converge.

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$$\begin{aligned} \lim_{n\to\infty} u_n &= u \\ \lim_{n\to\infty} v_n &= v \end{aligned}$$

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These limits always exist on the extended real line $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, +\infty\}$.

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Thus, $\liminf a_n = -1$ and $\limsup a_n = +1$.

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Some theorems

- For a real sequence (a_n) the limit $\lim a_n$ exists iff $\liminf a_n = \limsup a_n$. (▸ ?)
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This gives us the simplest notion of convergence of a sequence of functions.

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- In general, the value of $k_0(z, \epsilon)$ depends on the point z as well as the value of ϵ.

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Case 2 : The proof for $\lim a_n = -\infty$ is similar.

$\limsup a_n = \liminf a_n \Longleftarrow a_n$ converges

Go Back!

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$\because \epsilon > 0$ is arbitrary, we must have

$$\liminf a_n = \limsup a_n = a = \lim a_n$$

$\limsup a_n = \liminf a_n \implies a_n$ converges

Go Back!

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Left as an exercise!!