Sequences of functions

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MA211, Lecture 14

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Given a sequence of functions f_k, the values taken by the functions at a fixed value of $z \in S \subset \mathbb{C}$ yield a complex sequence $f_k(z)$.
This gives us the simplest notion of convergence of a sequence of functions.

Sequence of functions - pointwise convergence

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- This means that $\forall z \in S$ and $\forall \epsilon > 0$, $\exists k_0(z,\epsilon) \in \mathbb{N}$:

$$k > k_0(z,\epsilon) \implies |f_k(z) - f(z)| < \epsilon \ \forall z \in S.$$

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$$k > k_0(z,\epsilon) \implies |f_k(z) - f(z)| < \epsilon \ \forall z \in S.$$

- In general, the value of $k_0(z,\epsilon)$ depends on the point z as well as the value of ϵ.

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$f_k : S = \{z : |z| < 1\} \to \mathbb{C}, \; z \mapsto z^k, \; k \in \mathbb{N}.$

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The sequence converges pointwise to the zero function on S.

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To check this, we need to show that

$$\forall \epsilon > 0, \quad \exists k_0(z, \epsilon) \in \mathbb{N} \ :$$

$$k > k_0(z, \epsilon) \implies \left|z^k - 0\right| < \epsilon$$

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Note : $k_0(z,\epsilon)$ depends on z.

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Thus we can choose

$$k_0(z, \epsilon) = \max \left\{ \text{int} \left(\frac{\log(\epsilon)}{\log(r)} \right), 1 \right\}$$

- a value independent of z!

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- In general , the starting point of the "tail" depends on z.
- In some cases, however, we can choose a starting point that is the same for all z.
- The second demand is, of course, a stronger one than that needed for pointwise convergence.
- As we will see, it is also often more useful.

Uniform convergence

A sequence f_k is defined to converge **uniformly** to $f(z)$ on S if $\forall \epsilon > 0$, $\exists k_0(\epsilon) \in \mathbb{N}$:

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An alternative definition :

A sequence of functions f_k converges uniformly to f on $S \subset \mathbb{C}$ if $\forall \epsilon > 0$, $\exists k_0(\epsilon) \in \mathbb{N}$:

$$k > k_0(\epsilon) \implies \sup_{z \in S} |f_k(z) - f(z)| < \epsilon$$

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- converges uniformly to 0 on $S_r = \{z : |z| \le r < 1\}$.
- converges pointwise, but not uniformly, to 0 on $S = \{z : |z| < 1\}$.
- does not converge on the closed unit disc.

Examples

The sequence $f_k(z) = \frac{1}{1+kz}$ converges uniformly in $S = \{z : |z| \geq 2\}$ to the identically zero function.

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Thus, for any $\epsilon > 0$ we can choose $k_0(\epsilon) = \text{int}\left(\frac{1}{\epsilon}\right)$.

Examples

Indeed the sequence $f_k(z) = \frac{1}{1+kz}$ converges pointwise on $S = \mathbb{C} \setminus \left\{ -\frac{1}{n} : n \in \mathbb{N} \right\}$ to the function

$$f(z) = \begin{cases} 1 & \text{for } z = 0 \\ 0 & \text{for } z \in S - \{0\} \end{cases}$$

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This convergence is not uniform on S.

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The sequence converges uniformly to $\dfrac{1}{1-z}$ on S_r.

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To prove that the convergence is uniform on S_r note that

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Thus all we need is to ensure that $\frac{r^k}{1-r} < \epsilon$!

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So we can choose

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Note that the convergence is not uniform on the open unit disc.

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If f_k converges uniformly to f on S, then f is continuous if all the f_k are continuous.

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After Fourier and Abel had found counterexamples in the context of Fourier series, Dirichlet analysed Cauchy's proof and found out the error - thus the concept of uniform convergence was born!

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- $\because$ f_k is continuous on S, we can choose $\delta > 0$ such that

$$|z - z_0| < \delta \implies |f_k(z) - f(z_0)| < \frac{\epsilon}{3}$$

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&\quad + |f_k(z_0) - f(z_0)| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon
\end{aligned}$$

Uniform Cauchy sequences

Let (f_k) be a sequence of functions $S \subset \mathbb{C} \to \mathbb{C}$.
Then (f_k) is said to be **uniformly Cauchy** on S if

$$\forall \epsilon > 0, \; \exists N \in \mathbb{N} \;\; :$$

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Theorem

A sequence of functions f_k converges uniformly to a function f iff it is uniformly Cauchy.

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Theorem

A sequence of functions f_k converges uniformly to a function f iff it is uniformly Cauchy.

The proof that a uniformly convergent sequence of functions is uniformly Cauchy is trivial.

Uniform Cauchy $\implies$ uniform convergence

Let f_k be uniformly Cauchy on S.

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What we need to show is that the convergence of f_k to f is *uniform*.

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Thus

$$|f_k(z) - f(z)| < \epsilon$$