Complex Series

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MA211, Lecture 15

► Given a complex sequence (z_n) the formal expression

$$\sum_{k=1}^{\infty} z_k = z_1 + z_2 + \ldots + z_n + \ldots$$

is called an **infinite series** and z_1, z_2 etc. are called **terms** of the series.

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▶ The sum of the first *n* terms of the series

$$S_n = \sum_{k=1}^n z_k = z_1 + z_2 + \ldots + z_n$$

is called the *n*-th partial sum of the series.



► The infinite series is said to **converge to** *S* if the limit

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- ▶ In this case we write $S = \sum_{k=1}^{30} z_k$.
- ▶ If a series does not converge it is called divergent.
- Note that changing a *finite number* of terms in a series has no effect on its convergence or divergence!



Theorem

If the series $\sum_{k=1}^{\infty} z_k$ converges, then $\forall \epsilon > 0$, $\exists N(\epsilon) \in \mathbb{N}$ such that

$$n \geq m > N(\epsilon) \implies |z_m + z_{m+1} + \dots z_n| < \epsilon$$

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Proof.

 $\therefore S_n$ converges, it is a Cauchy sequence.

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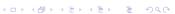
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$$|z_n - 0| < \epsilon$$

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For $|z| \ge 1$, the series diverges from the last theorem.

Absolute convergence

A series is called **absolutely convergent** if the real series of magnitudes $\sum_{k=1}^{\infty} |z_k|$ exists.

Example:

The geometric series $\sum_{k=0}^{\infty} z^k$ converges absolutely for |z| < 1.

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 z_n is a Cauchy sequence. It is convergent!

An application: the complex exponential

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The series S converges absolutely, and hence converges for all $z \in \mathbb{C}$.

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$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

is convergent.

The series

▶ Why?

An absolutely convergent series is convergent, the converse is not necessarily true!

The series

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

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▶ Why?

But

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▶ Why?

S is convergent, but not absolutely convergent.

Connection with real series

Theorem

Let
$$z_n = x_n + iy_n$$
 and $S = U + iV$. Then

$$S = \sum_{n=1}^{\infty} z_n$$

iff

$$U = \sum_{n=1}^{\infty} x_n, \ V = \sum_{n=1}^{\infty} y_n$$

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The proof follows trivially from the corresponding result for complex sequences.

$$\sum_{n=1}^{\infty} \frac{1 + i n (-1)^n}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{1 + in(-1)^n}{n^2}$$
We have

$$\sum_{n=1}^{\infty} \frac{1 + i n (-1)^n}{n^2} = \sum_{n=1}^{\infty} \left[\frac{1}{n^2} + i \frac{(-1)^n}{n} \right]$$

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are convergent.

$$\sum_{n=1}^{\infty} \frac{1+\mathfrak{i}n(-1)^n}{n^2}$$

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The complex series above is convergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n + \mathfrak{i}}{n}$$

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We know that though the real series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is

convergent, the real series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

The complex series above is divergent.



Some results for real series

• In view of the connection of complex series with real ones, as well as the fact that absolutely convergent series play a very important role in our course, we will now review some results from real analysis on series.

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- ▶ **Corollary**: If $0 \le a_n \le b_n$ for every $n \in \mathbb{N}$, and if $\sum a_n$ diverges, then $\sum b_n$ diverges.
- ▶ Limit comparison test : If $0 \le a_n \le b_n$ for every $n \in \mathbb{N}$, and if $0 < \lim \frac{a_n}{b_n} = c < \infty$, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

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The sum $\sum \frac{2n^2}{3n^3+1}$ diverges.

Proof.

$$\lim_{n \to \infty} \left(\frac{2n^2}{3n^3 + 1} \right) / \left(\frac{1}{n} \right)$$

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Since $\sum \frac{1}{n}$ diverges, so does $\sum \frac{2n^2}{3n^3+1}$



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$$\lim_{n\to\infty} \left(\frac{3n+5}{n2^n}\right) / \left(\frac{1}{2^n}\right) = \lim_{n\to\infty} 3 + \frac{5}{n} = 3 > 0$$

Since the geometric series $\sum \frac{1}{2^n}$ converges, so does

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 - 1. absolutely converges is $\alpha < 1$;
 - 2. diverges if $\alpha > 1$ or $+\infty$.
 - ▶ The root test fails if $\alpha=1$. In this case the series may either converge or diverge. ▶ Proof
- ▶ d'Alembert's ratio test : Let $\sum a_n$ be a series of nonzero terms. Then the series $\sum a_n$

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- ▶ d'Alembert's ratio test : Let $\sum a_n$ be a series of nonzero terms. Then the series $\sum a_n$
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$$\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq 1 \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|.$$

In this case the ratio test fails.



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From the ratio test, S converges absolutely for all values of x.

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From the ratio test, S converges absolutely for |x| < 1 and diverges for |x| > 1.

Comparison test:

Let $\sum_{n=1}^{\infty} M_n$ be a convergent real sequence of nonnegative terms and $|z_n| < M_n$ for all n then

$$\sum_{n=1}^{\infty} z_n$$

converges absolutely.

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Proof.

Using the comparison test for real series we see that $\sum_{n=1}^{\infty} |z_n|$ is convergent. Thus the series is absolutely convergent, and thus convergent.

Ratio test:

Given the series $\sum_{n=1}^{\infty} \zeta_n$ of nonzero terms, let

$$\lim \sup \left| \frac{\zeta_{n+1}}{\zeta_n} \right| = L$$

Then the series is absolutely convergent if L < 1 and divergent if L > 1.

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Proof.

Use the ratio test for real series to show that the real series of magnitudes $\sum |\zeta_n|$ converges if L < 1. Thus $\sum \zeta_n$ is absolutely convergent.

Root test:

Given the series $\sum_{n=1}^{\infty} \zeta_n$, let

$$\limsup |\zeta_n|^{1/n} = L$$

Then the series is absolutely convergent for L < 1 and divergent for L > 1.

The harmonic series
$$\sum_{n=1}^{\infty} \frac{1}{n}$$
 diverges.

$$S_n = \sum_{k=1}^{\infty} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n}$$

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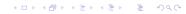
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Let
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Thus (s_n) is a bounded and monotone sequence - it converges.

 \therefore The series $\sum_{k=1}^{\infty} a_k$ converges.

◆ Go Back!



Suppose
$$\lim \frac{a_n}{b_n} = c > 0$$
.

Suppose $\lim \frac{a_n}{b_n} = c > 0$. Thus $\exists N \in \mathbb{N}$:

$$n > N \implies \left| \frac{a_n}{b_n} - c \right| < \frac{c}{2}$$

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Thus for $n > N(\epsilon)$, $0 \le |a_n| < (\alpha + \epsilon)^n$. $0 < \alpha + \epsilon < 1$ the geometric series $\sum (\alpha + \epsilon)^n$ is convergent.

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From the term comparison test, we see that $\sum a_n$ converges.

For the proof we need the lemma

$$\liminf \left| \frac{a_{n+1}}{a_n} \right| \le \liminf |a_n|^{1/n}$$

$$\leq \limsup |a_n|^{1/n} \leq \limsup \left| \frac{a_{n+1}}{a_n} \right|$$

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Proof of the lemma as well as the rest of the proof is left as an exercise.

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Implications:

▶ If $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$ exists, then so does $\lim_{n \to \infty} \left| a_n \right|^{1/n}$

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Implications:

- ▶ If $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$ exists, then so does $\lim_{n \to \infty} \left| a_n \right|^{1/n}$
- ▶ If the root test fails, the ratio test will fail too.

