Complex Series

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MA211, Lecture 15

Series

- Given a complex sequence (z_n) the formal expression

$$\sum_{k=1}^{\infty} z_k = z_1 + z_2 + \ldots + z_n + \ldots$$

is called an **infinite series** and z_1, z_2 etc. are called **terms** of the series.

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- The sum of the first n terms of the series

$$S_n = \sum_{k=1}^{n} z_k = z_1 + z_2 + \ldots + z_n$$

is called the n-th partial sum of the series.

Series

- The infinite series is said to **converge to** S if the limit

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} z_k$$

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- In this case we write $S = \displaystyle\sum_{k=1}^{\infty} z_k$.
- If a series does not converge it is called **divergent**.
- Note that changing a *finite number* of terms in a series has no effect on its convergence or divergence!

Cauchy theorem for series

Theorem

If the series $\sum_{k=1}^{\infty} z_k$ *converges, then* $\forall \epsilon > 0$, $\exists N(\epsilon) \in \mathbb{N}$ *such that*

$$n \geq m > N(\epsilon) \implies |z_m + z_{m+1} + \ldots z_n| < \epsilon$$

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$$|z_n - 0| < \epsilon$$

Geometric series

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For $|z| \geq 1$, the series diverges from the last theorem.

Absolute convergence

A series is called **absolutely convergent** if the real series of magnitudes $\sum_{k=1}^{\infty} |z_k|$ exists.

Example :
The geometric series $\sum_{k=0}^{\infty} z^k$ converges absolutely for $|z| < 1$.

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z_n is a Cauchy sequence. It is convergent!

An application : the complex exponential

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The series S converges absolutely, and hence converges for all $z \in \mathbb{C}$.

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The series

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

is convergent.

Why?

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$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

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S is convergent, but not absolutely convergent.

Connection with real series

Theorem

Let $z_n = x_n + \mathrm{i} y_n$ *and* $S = U + \mathrm{i} V$. *Then*

$$S = \sum_{n=1}^{\infty} z_n$$

iff

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The proof follows trivially from the corresponding result for complex sequences.

Applications

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We have

$$\sum_{n=1}^{\infty} \frac{1 + \mathfrak{i} n(-1)^n}{n^2} = \sum_{n=1}^{\infty} \left[\frac{1}{n^2} + \mathfrak{i} \frac{(-1)^n}{n} \right]$$

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The complex series above is convergent. □

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Some results for real series

• In view of the connection of complex series with real ones, as well as the fact that absolutely convergent series play a very important role in our course, we will now review some results from real analysis on series.

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Tests of convergence for real series

- **Term comparison test :** If $0 \leq a_n \leq b_n$ for every $n \in \mathbb{N}$, and if $\sum b_n$ converges, then $\sum a_n$ converges. ▸ Proof

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- **Corollary :** If $0 \leq a_n \leq b_n$ for every $n \in \mathbb{N}$, and if $\sum a_n$ diverges, then $\sum b_n$ diverges.
- **Limit comparison test :** If $0 \leq a_n \leq b_n$ for every $n \in \mathbb{N}$, and if $0 < \lim \frac{a_n}{b_n} = c < \infty$, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge. ▸ Proof

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The sum $\sum \frac{2n^2}{3n^3+1}$ diverges.

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$$\lim_{n\to\infty} \left(\frac{2n^2}{3n^3+1}\right) / \left(\frac{1}{n}\right)$$

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$$\lim_{n\to\infty} \left(\frac{2n^2}{3n^3+1}\right) / \left(\frac{1}{n}\right)$$

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$$= \lim_{n\to\infty} \frac{2n^3}{3n^3+1} = \lim_{n\to\infty} \frac{2}{3+\frac{1}{n^3}}$$

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$$\lim_{n\to\infty}\left(\frac{2n^2}{3n^3+1}\right)/\left(\frac{1}{n}\right)$$

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Since $\sum \frac{1}{n}$ diverges, so does $\sum \frac{2n^2}{3n^3+1}$ □

Applications

The sum $\sum \frac{3n+5}{n2^n}$ converges.

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The sum $\sum \frac{3n+5}{n2^n}$ converges.

Proof.

$$\lim_{n\to\infty} \left(\frac{3n+5}{n2^n}\right) / \left(\frac{1}{2^n}\right) = \lim_{n\to\infty} 3 + \frac{5}{n} = 3 > 0$$

Since the geometric series $\sum \frac{1}{2^n}$ converges, so does $\sum \frac{2n^2}{3n^3+1}$ □

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$$\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq 1 \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|.$$

In this case the ratio test fails.

▸ Proof

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From the ratio test, S converges absolutely for $|x| < 1$ and diverges for $|x| > 1$.

Comparison test :
Let $\sum_{n=1}^{\infty} M_n$ be a convergent real sequence of nonnegative terms and $|z_n| < M_n$ for all n then

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Proof.
Using the comparison test for real series we see that $\sum_{n=1}^{\infty} |z_n|$ is convergent. Thus the series is absolutely convergent, and thus convergent. □

Ratio test :

Given the series $\sum_{n=1}^{\infty} \zeta_n$ of nonzero terms, let

$$\limsup \left| \frac{\zeta_{n+1}}{\zeta_n} \right| = L$$

Then the series is absolutely convergent if $L < 1$ and divergent if $L > 1$.

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Proof.

Use the ratio test for real series to show that the real series of magnitudes $\sum |\zeta_n|$ converges if $L < 1$.Thus $\sum \zeta_n$ is absolutely convergent. □

Root test :

Given the series $\sum_{n=1}^{\infty} \zeta_n$, let

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$\therefore$ The series $\sum_{k=1}^{\infty} a_k$ converges.

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Thus for $n > N(\epsilon), 0 \leq |a_n| < (\alpha + \epsilon)^n$.

$\because 0 < \alpha + \epsilon < 1$ the geometric series $\sum(\alpha + \epsilon)^n$ is convergent.

Proof of Cauchy root test

Suppose $\alpha = \limsup |a_n|^{1/n} < 1$. Select $\epsilon > 0$ such that $\alpha + \epsilon < 1$.
$\because \lim_{n\to\infty}\left(\sup\left\{|a_n|^{1/n}\right\}\right) = \alpha$, $\exists N(\epsilon) \in \mathbb{N}$:

$$n > N(\epsilon) \implies \alpha - \epsilon < \sup\left\{|a_n|^{1/n}\right\} < \alpha + \epsilon$$

Thus for $n > N(\epsilon), 0 \leq |a_n| < (\alpha+\epsilon)^n$.
$\because 0 < \alpha + \epsilon < 1$ the geometric series $\sum(\alpha+\epsilon)^n$ is convergent.
From the term comparison test, we see that $\sum a_n$ converges.

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Proof of d'Alembert's ratio test

For the proof we need the lemma

$$\liminf \left| \frac{a_{n+1}}{a_n} \right| \leq \liminf |a_n|^{1/n}$$

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Proof of the lemma as well as the rest of the proof is left as an exercise.

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Implications :

- If $\lim \left|\frac{a_{n+1}}{a_n}\right|$ exists, then so does $\lim |a_n|^{1/n}$

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Implications :

- If $\lim \left|\frac{a_{n+1}}{a_n}\right|$ exists, then so does $\lim |a_n|^{1/n}$
- If the root test fails, the ratio test will fail too.

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