Power Series

Ananda Dasgupta

MA211, Lecture 16

Power Series

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- c_1, c_2 etc. are called its **coefficients**.
- If the power series converges at all points of a region $S \subset \mathbb{C}$ then it defines a function

$$f : S \to \mathbb{C}, \ f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

called a **power series function**.

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- We will see later that all complex holomorphic functions are analytic, too!
- In complex analysis the terms **analytic** and **holomorphic**, though defined differently - are actually equivalent.
- Many textbooks, use the terms interchangably!

Examples

- The function $\frac{1}{1-z}$ is analytic at $z = 0$ as can be seen from the power series expansion
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- However it is analytic on $\mathbb{C} \setminus \{0\}$.

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- In particular, its power series expansion at $z = 0$ is given by

$$g(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$$

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- The function $\exp\left(\frac{1}{z}\right)$ is not analytic at $z = 0$.
- As its expansion in powers of z^-1,

$$\exp\left(\frac{1}{z}\right) = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots$$

shows, $g(z) = z^n f(z)$ is not analytic for any $n \in \mathbb{N}$.

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- **Branch points** : a branch of a multifunction is not continuous and hence not analytic at all points on its branch cut.

Disc of convergence - a theorem :

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- the point z_0 only.
- the disc $D_\rho (z_0) = \{z \ : \ |z - z_0| < \rho\}$ along with part (all, some or none) of its boundary $C_\rho (z_0) = \{z \ : \ |z - z_0| = \rho\}$

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For the two extreme cases we define ρ to be 0 and ∞, respectively.

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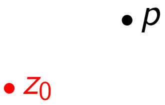
ρ is called the **radius of convergence** of the power series.

Disc of convergence - an informal view :



Consider a power series

$$\sum_{n=0}^{\infty} c_n (z - z_0)^n$$

centered at z_0.

Suppose that we know that it converges at the point p.

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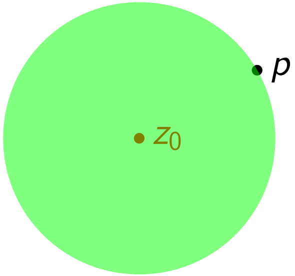

Then the series converges absolutely at all points closer to z_0 than p, *i.e.* all points of the open disc centered at z_0 whose boundary passes through p :

$$|z - z_0| < |p - z_0|$$

Why?

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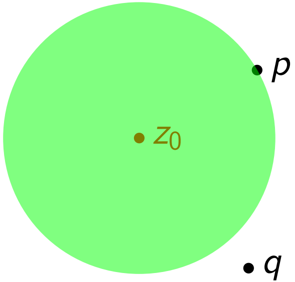
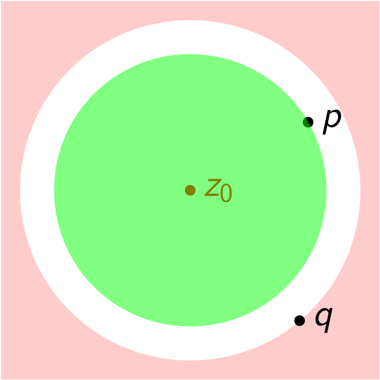


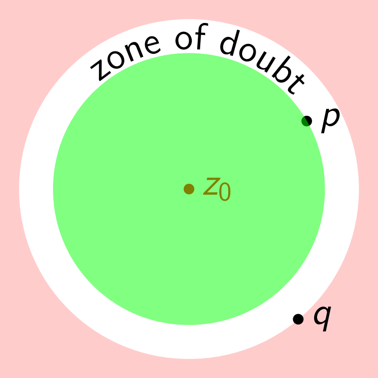
Next suppose we find that the series diverges at a point q, further away from z_0 than p.

Disc of convergence - an informal view :



Then the series must diverge everywhere further away from z_0.

Disc of convergence - an informal view :



As yet, we do not know the nature of the series in the "zone of doubt" :

$$|p - z_0| < |z - z_0| < |q - z_0|$$

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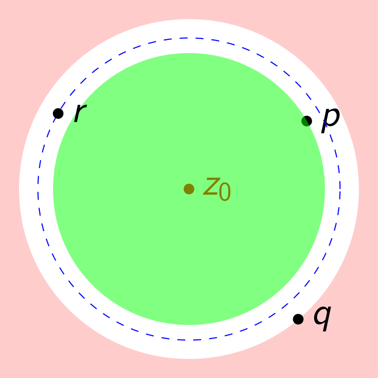

We next look at the convergence of the series at a point r midway inside the “zone of doubt”.

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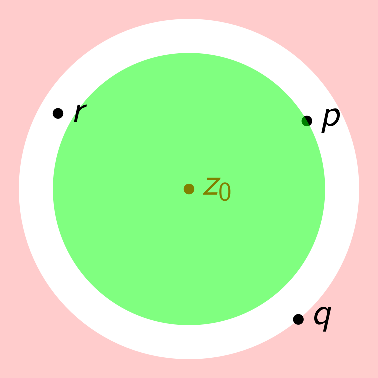

The series may either converge or diverge at r.

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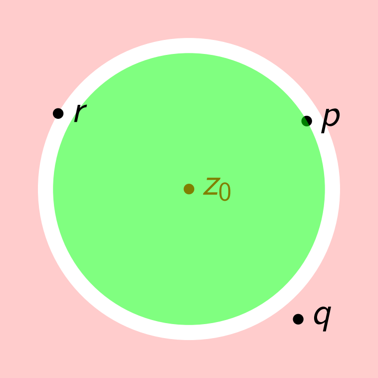

The series may either converge or diverge at r. Either way the "zone of doubt" shrinks.

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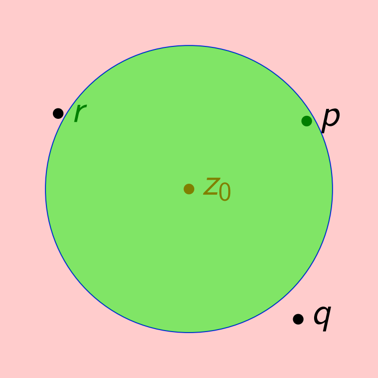

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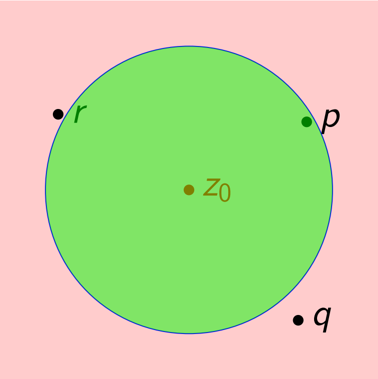

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The power series converges absolutely everywhere inside the disc of convergence :

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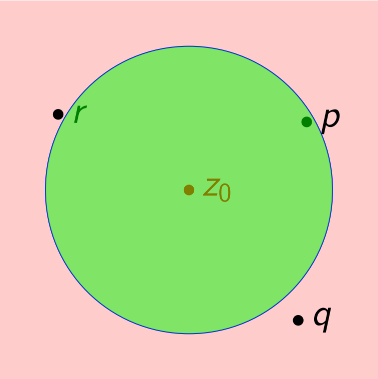

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The power series converges absolutely everywhere inside the disc of convergence :

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It diverges everywhere outside!

Theorem on convergence - formal proof :

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Given the series $\sum_{n=0}^{\infty} \zeta_n$, let

$$\limsup |\zeta_n|^{1/n} = L$$

Then the series is absolutely convergent for $L < 1$ and divergent for $L > 1$.

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Also $|z - z_0| > \rho \implies$ the power series diverges.

Formulas for the radius of convergence

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- **d'Alembert's ratio formula :**

$$\rho = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|$$

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Thus, the limit

$$\lim_{n\to\infty} |c_n|^{1/n} = \lim_{n\to\infty} \frac{n+2}{3n+1}$$

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Here $c_n = \left(\frac{n+2}{3n+1} \right)^n \implies |c_n|^{1/n} = \frac{n+2}{3n+1}.$

Thus, the limit

$$\lim_{n\to\infty} |c_n|^{1/n} = \lim_{n\to\infty} \frac{n+2}{3n+1} = \frac{1}{3}$$

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The disc of convergence of this power series function is

$$\{z \ : \ |z-4| < 3\}$$

Examples

Consider the power series function :

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The disc of convergence has a radius of $\frac{1}{5}$.

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Thus the exponential function is analytic on the whole complex plane!

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This can be made smaller than any $\epsilon > 0$ for large enough $m \implies$ the series is Cauchy at z!.

Back!