Power Series contd.

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MA211, Lecture 17

Differentiating power series

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$$\sum_{n=0}^{N} c_n (z - z_0)^n = c_0 + c_1 (z - z_0) + \ldots + c_N (z - z_0)^N$$

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Rediscovering the exponential function

The familiar function exp $\mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

can be generalized to the complex domain by directly looking at the series.

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Thus it defines a *unique* analytic function that matches e^x on the real line.

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We *define* the complex exponential function by

$$f : \mathbb{C} \to \mathbb{C}, \quad \frac{df}{dz} = f(z), \quad f(0) = 1$$

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$$f(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots + c_n z^n + \ldots$$

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We have once again recovered the exponential function.

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Assume a power series solution of the form

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$$0 = f'(z) - f(z) = \sum_{n=0}^{\infty} n c_n z^{n-1} - \sum_{n=0}^{\infty} c_n z^n$$

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which implies $(n+1)c_{n+1} - c_n = 0$ as before!

A more complicated example

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$$f''(z) - 2zf'(z) + 2\nu f(z) = 0$$

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Substituting these in Hermite's equation we have

$$\sum_{n=0}^{\infty}(n+1)(n+2)c_{n+2}z^{n}-2z\sum_{n=0}^{\infty}nc_nz^{n-1}+2\nu\sum_{n=0}^{\infty}c_nz^{n}=0$$

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This recursion relation allows us to calculate

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General solution : $f(z) = c_0\phi_0(z) + c_1\phi_1(z)$

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The answer is - no!

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Thus the only power series that will satisfy our equation is identically zero - *the trivial solution*!

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- The answer to all these is - YES!

The arithmatic of power series

Let $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ and $g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$ have radii of convergence ρ_1 and ρ_2, respectively.

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$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

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A theorem on differentiation of power series :

Suppose a power series function $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$ has a radius of convergence $\rho > 0$. Then

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We need only to prove this for $k = 1$. Repeated application then gives the general case!

Uniqueness of power series expansions - a simple corollary :

Let the function $f : S \subset \mathbb{C} \to \mathbb{C}$ have two power series expansions centered around the same point z_0 :

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} b_n (z - z_0)^n .$$

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Given a power series function

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\end{aligned}$$

$\Longrightarrow$ the radius of convergence of the power series for $g(z)$ is the same as that for $f(z)$.

Differentiating power series - proof :

Define

$$S_j(z) = \sum_{n=0}^{j} c_n (z - z_0)^n$$

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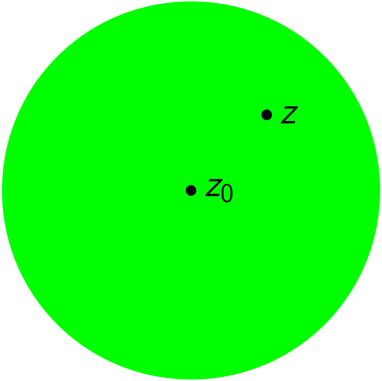
Differentiating power series - proof :



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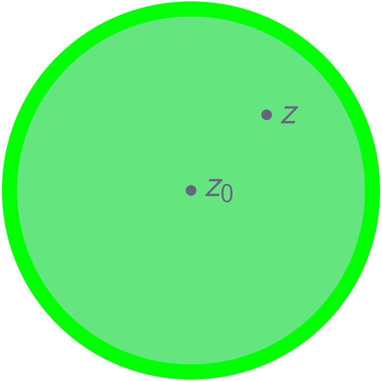

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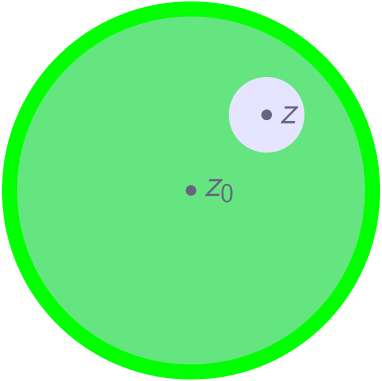

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Let $z \in D_\rho(z_0)$ and choose $\epsilon > 0$.

Choose $r < \rho$ so that $z \in D_r(z_0)$and $\delta_1 > 0$ such that

$$D_{\delta_1}(z) \subset D_r(z_0) \subset D_\rho(z_0)$$

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&\quad - g(z) \\
&= \left[\frac{S_j(z') - S_j(z)}{z' - z} - S_j'(z)\right] \\
&\quad + \left[S_j'(z) - g(z)\right] \\
&\quad + \left[\frac{R_j(z') - R_j(z)}{z' - z}\right]
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$$\underbrace{\frac{f(z') - f(z)}{z' - z} - g(z)}_{A} = \underbrace{\left[\frac{S_j(z') - S_j(z)}{z' - z} - S_j'(z)\right]}_{B}$$

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Differentiating power series - proof :

$$\underbrace{\frac{f(z')-f(z)}{z'-z}-g(z)}_{A} = \underbrace{\left[\frac{S_j(z')-S_j(z)}{z'-z}-S_j'(z)\right]}_{B} + \underbrace{\left[S_j'(z)-g(z)\right]}_{C} + \underbrace{\left[\frac{R_j(z')-R_j(z)}{z'-z}\right]}_{D}$$

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We need to show that by taking z' sufficiently close to z we can make each term smaller than $\frac{\epsilon}{3} > 0$.

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Term D :

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$$= \sum_{n=j+1}^{\infty} |c_n| \left|\frac{s^n - t^n}{s-t}\right|, \quad s = z' - z_0, \quad t = z - z_0$$

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$\because r < \rho$, the series $\sum_{n=0}^{\infty} |c_n| \, n r^{n-1}$ converges.

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$|D|$, the magnitude of its tail, can be made as small as you may want!

$\forall \epsilon > 0$ we can choose $N_1(\epsilon) \in \mathbb{N}$:

$$j > N_1(\epsilon) \implies |D| < \frac{\epsilon}{3}$$

Differentiating power series - proof :

Term C :

$\because S_j'(z) = \sum_{n=0}^{j} nc_n (z - z_0)^{n-1}$, and $g(z) = \sum_{n=0}^{\infty} nc_n (z - z_0)^{n-1}$ we have

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$$j > N_2(\epsilon) \implies |C| = \left| S_j'(z) - g(z) \right| < \frac{\epsilon}{3}$$

Differentiating power series - proof :

Term B :

So far, we have

$$j > N_1(\epsilon) \implies |D| < \frac{\epsilon}{3}$$
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Choose $N(\epsilon) = \max\{N_1(\epsilon), N_2(\epsilon)\}$, set $j > N(\epsilon)$.

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$$\because \lim_{z' \to z} \frac{S_j(z') - S_j(z)}{z' - z} = S_j'(z)$$

$$|z' - z| < \delta_2 \implies |B| = \left| \frac{S_j(z') - S_j(z)}{z' - z} - S_j'(z) \right| < \frac{\epsilon}{3}$$

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