Taylor and Laurent Series

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MA211, Lecture 18

Laurent series

A doubly infinite series $\sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$, called the **Laurent series**, is defined by

$$\sum_{n=-\infty}^{\infty} c_n (z - z_0)^n \equiv \sum_{n=-\infty}^{-1} c_n (z - z_0)^n + \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

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- If the set $\{n \in \mathbb{N} : c_{-n} \neq 0\}$ is finite, the Laurent series function has a pole at z_0.
- Otherwise, the point z_0 is an essential singularity.

Two important theorems

If a function is holomorphic in an open disc $D_r(z_0) = \{z : |z - z_0| < r\}$, then it can be expanded in a Taylor series centered at z_0 which has a radius of convergence $\rho \geq r$.

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If a function is holomorphic in an open annulus $A_{r_1,r_2}(z_0) = \{z : r_1 < |z - z_0| < r_2\}$, then it can be expanded in a Laurent series centered at z_0 which converges in an annulus $A_{\rho_1,\rho_2}(z_0) \supset A_{r_1,r_2}(z_0)$.

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The proofs of these theorem depend on the Cauchy integral formula that we will derive later.

Differentiating Laurent series

If a Laurent series function

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

converges in the annulus $A_{\rho_1,\rho_2}(z_0)$, then the term by term differentiated Laurent series

$$g(z) = \sum_{n=-\infty}^{\infty} nc_n (z - z_0)^{n-1}$$

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The idea behind the proof is the same as that for Taylor series.

Examples

Consider the function

$$\frac{3}{(1+z)(2-z)}$$

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We will find its Laurent series expansion in each of these three regions.

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$$\frac{1/2}{1-z/2} = \begin{cases} \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} & \text{for } |z| < 2 \\ \sum_{n=1}^{\infty} \frac{-2^{n-1}}{z^n} & \text{for } |z| > 2 \end{cases}$$

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$$\frac{3}{(1+z)(2-z)} = \sum_{n=0}^{\infty}\left[(-1)^n + \frac{1}{2^{n+1}}\right] z^n \text{ in } D$$

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$$\frac{3}{(1+z)(2-z)} = \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^n} \text{ in } A$$

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$$\frac{\cos z - 1}{z^4} = -\frac{1}{2!z^2} + \frac{1}{4!} - \frac{z^2}{6!} + \dots$$

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$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots$$

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$$\csc z \;=\; \frac{1}{z}\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \dots\right)^{-1}$$

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$$\begin{aligned}
\csc z &= \frac{1}{z}\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \dots\right)^{-1} \\
&= \frac{1}{z}\left[1 + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} + \dots\right)\right. \\
&\quad \left. + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} + \dots\right)^2 + \dots\right]
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&= \frac{1}{z}\left[1 + \left(\frac{1}{3!}\right)z^2 + \left(-\frac{1}{5!} + \frac{1}{(3!)^2}\right)z^4 + \dots\right]
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&= \frac{1}{z} + \frac{z}{6} + \frac{7z^3}{360} + \dots
\end{aligned}$$