Analyzing the Power Series method for linear ODEs

Ananda Dasgupta

MA211, Lecture 19

Analysis of ODEs in the complex plane

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- Despite that, looking at the behaviour of these equations in the *complex plane* allows us to draw important conclusions about their solutions!
- We will deal with the simple (but important) subclass of LODEs.

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- A general NOHLODE is given by

$$p_n(z)\frac{d^n f}{dz^n}+p_{n-1}(z)\frac{d^{n-1} f}{dz^{n-1}}+\ldots+p_1(z)\frac{df}{dz}+p_0(z)f(z)=0$$

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where $P_i(z)=\dfrac{p_i(z)}{p_n(z)},\ i=0,1,2,\ldots,n-1.$

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- The singularities of $P_i(z)$ dictate the singularities of the solutions.

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- We will first, however, take a look at FOHLODEs, where an analytic solution is possible.

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$\exists r_2 > r_1 > 0$ such that in an annular region $0 \leq r_1 < |z - z_0| < r_2$ we can write

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is analytic in the annulus $r_1 < |z - z_0| < r_2$.

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- If $P(z)$ has a stronger than first order pole at z_0, $f(z)$ has an essential singularity there!

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For a general NOHLODE :

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where the coffeicient functions $P_0(z), \ldots, P_{n-1}(z)$ are analytic in the annulus $r_1 < |z - z_0| < r_2$, we can write down n general independent solutions of the "standard" form

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- We can factor out the branch points.
- Can we also factor out the poles?

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- Otherwise, it has an essential singularity at $z = z_0$.
- If a given "standard" solution has only a pole at $z = z_0$ it can be written as
$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^{n+s}, \quad c_0 \neq 0$$
where s is called the **index** of this solution.

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- The condition for z_0 to be a regular singular point of a NOHLODE is that $(z - z_0)^{n-i} P_i(z)$ is analytic at $z = z_0$ for $i = 0, 1, 2, \ldots, n-1$.

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- We will provide a partially rigorous proof of this by construction!

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We seek **two** solutions of the kind

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The nature of the indicial equation depends on the relative sizes of 2, $p+1$ and q.

Analytic structure of solutions of SOHLODEs

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Both solutions *must* have essential singularities.

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While one solution is *formally* analytic - in most cases it will turn out to have a zero radius of convergence!

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$z = 0$ is a regular singular point.