Complex numbers
Algebra and Geometry

Ananda Dasgupta

MA211, Lecture 2

The algebra of complex numbers

- Consider the set $\mathbb{C}$ of ordered pairs of real numbers :

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- $\{\mathbb{C}, +\}$ is an Abelian group (?), with $(0, 0)$ being the additive identity.

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- $\{\mathbb{C}, +, *\}$ is a commutative ring (?) with identity.

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- $\{\mathbb{C}, +, *\}$ is a *field*(▸ ?).

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- $\mathbb{C}$ has a subset, R, that is isomorphic to $\mathbb{R}$.
- We will henceforth denote $(r, 0)$ by r.

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$$(a, b) = (a, 0) * (1, 0) + (b, 0) * (0, 1)$$

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$$(a,b) = (a,0)*(1,0) + (b,0)*(0,1) \equiv a + \mathfrak{i}b.$$

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- If $|z| = r$ and $\arg(z) = \theta$, we can write

$$z = r(\cos\theta + \imath\sin\theta) = r\angle\theta = re^{\imath\theta}$$

Some properties of complex conjugation

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- This extends to general functions!

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- Usually, $0 \leq \mathrm{Arg}(z) < 2\pi$, though $-\pi \leq \mathrm{Arg}(z) < \pi$ is also used.

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- $\sim$ is an equivalence relation.

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- Define multiplication on $\mathbb{R}[X]/K$ by
$$[P][Q] = [PQ].$$

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- Given $P \in \mathbb{R}[X]$, there exists unique polynomials $S, R \in \mathbb{R}[X]$ such that

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- We can label elements of $\mathbb{R}[X]/K$ by linear polynomials $a + bX$.

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- $[a + bX][c + dX] = [(ac - bd) + (bc + ad)X]$
- $\mathbb{R}[X]/\left(X^2 + 1\right)$ is isomorphic to $\mathbb{C}$!

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- We can use the same geometric representation for both!

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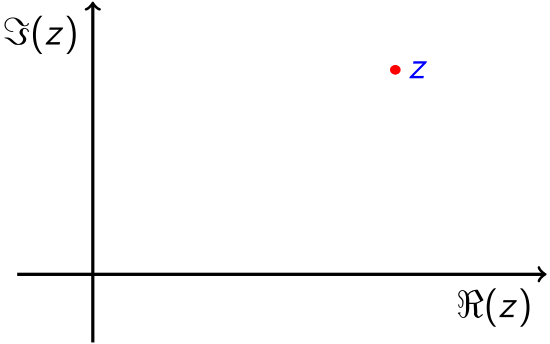

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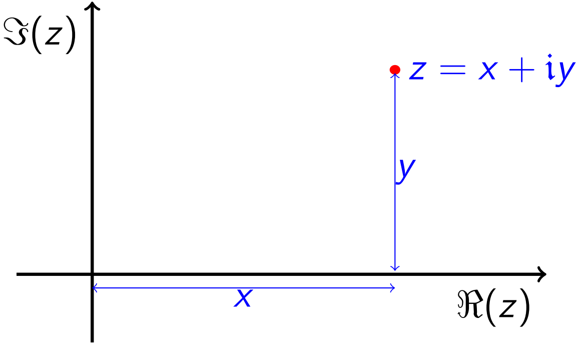

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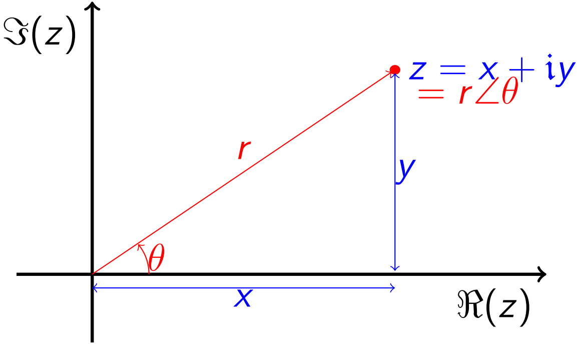

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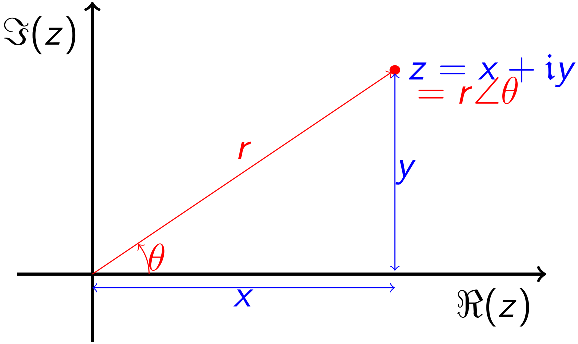

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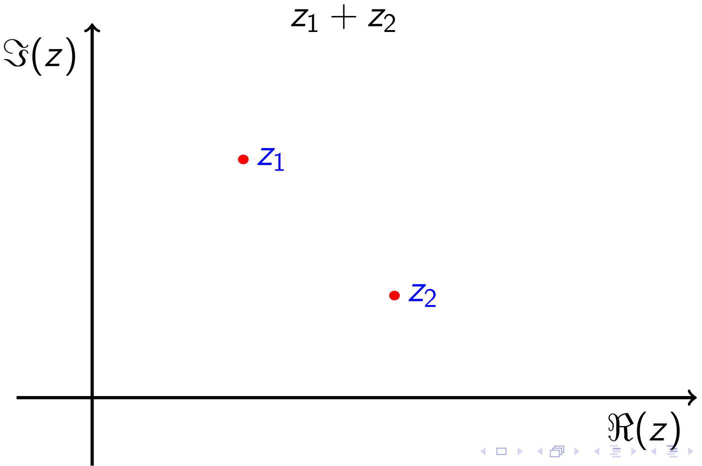

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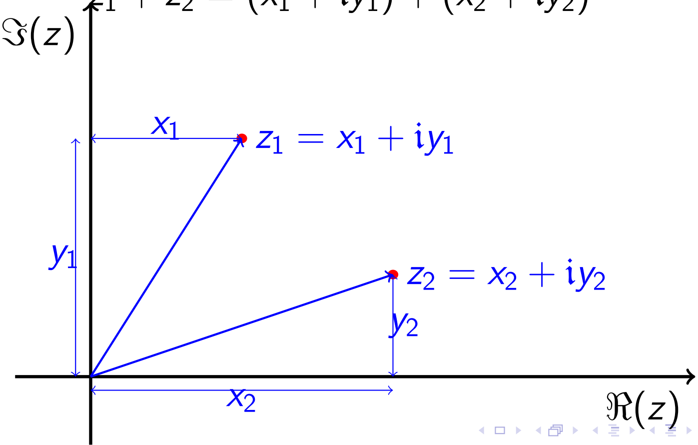

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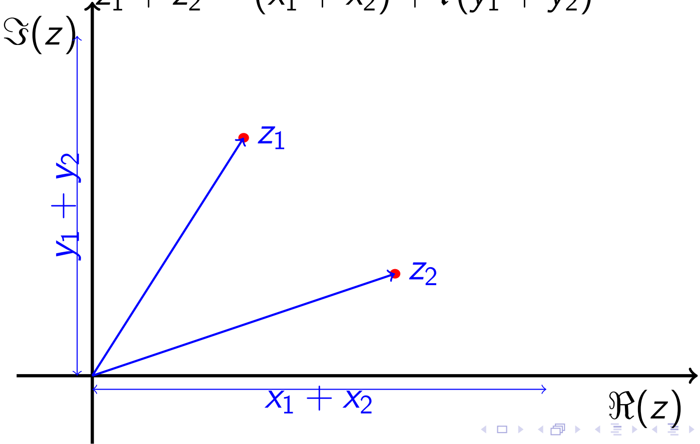

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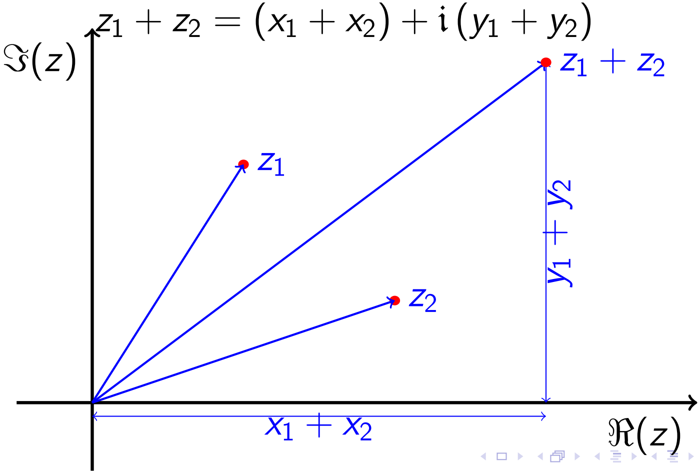

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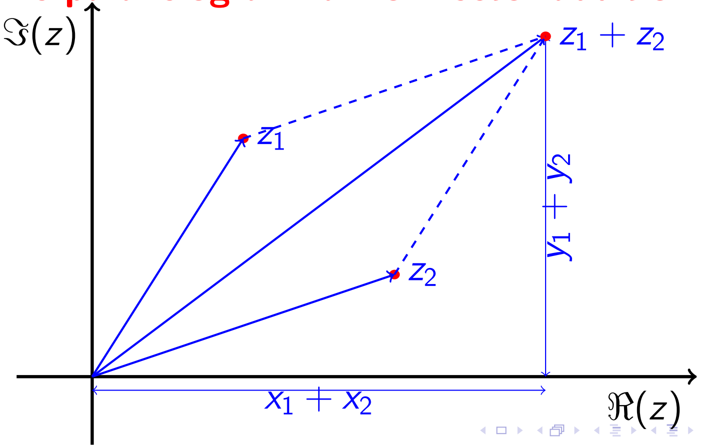
The parallelogram law of vector addition!



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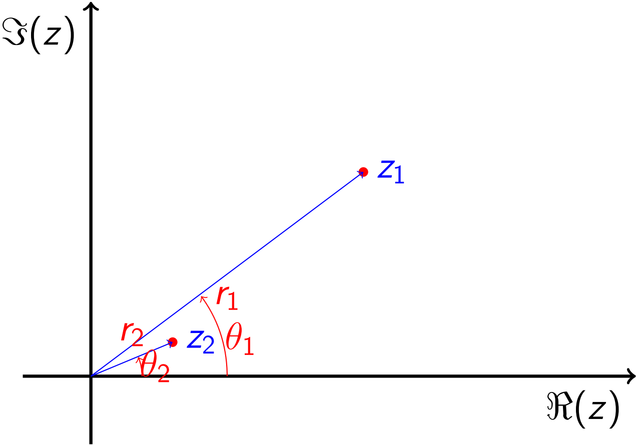
It helps to use the polar representation!

$$z_1 z_2 \;=\; r_1\angle\theta_1 \cdot r_2\angle\theta_2$$



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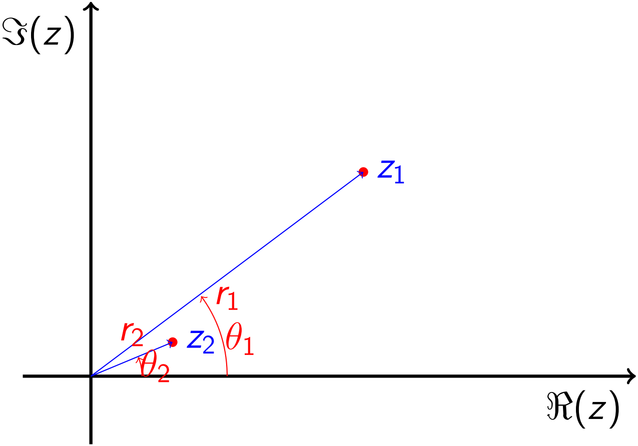
$$z_1 z_2 \;=\; r_1\left(\cos\theta_1 + \mathfrak{i}\sin\theta_1\right)\cdot r_2\left(\cos\theta_2 + \mathfrak{i}\sin\theta_2\right)$$



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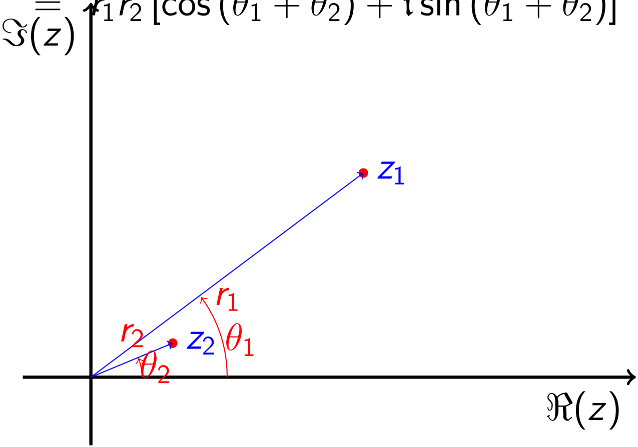

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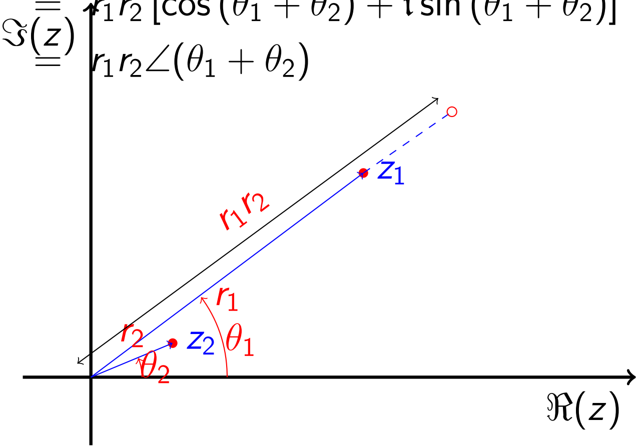

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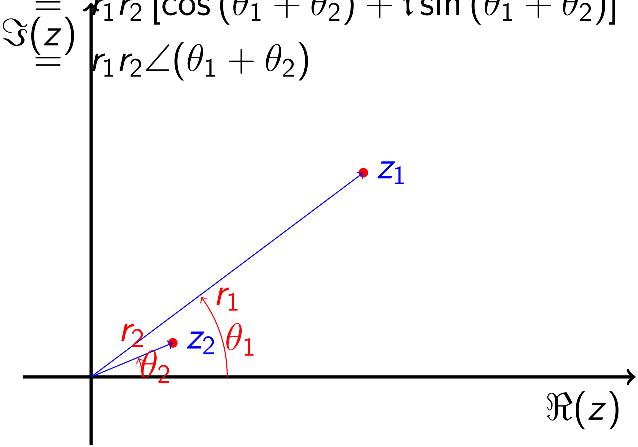

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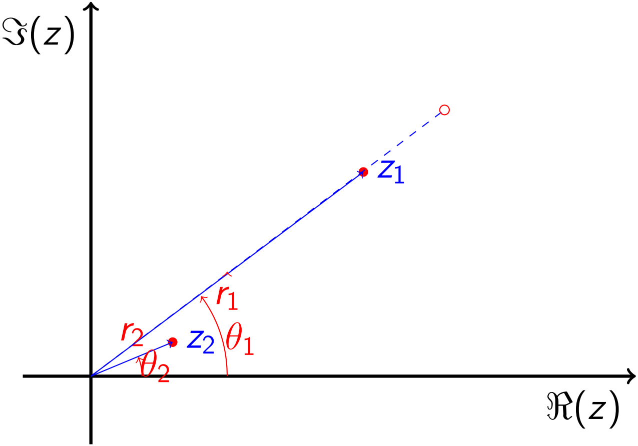

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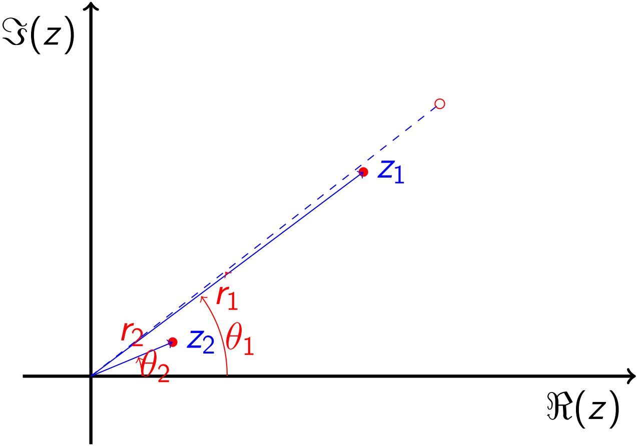

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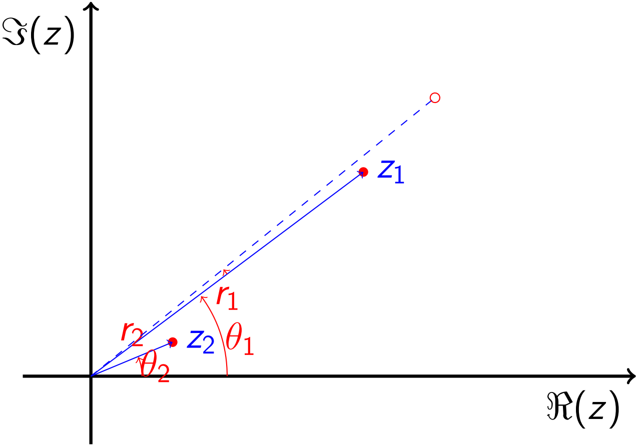

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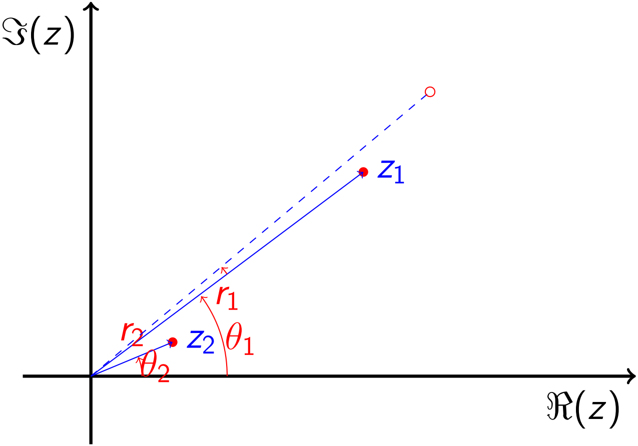

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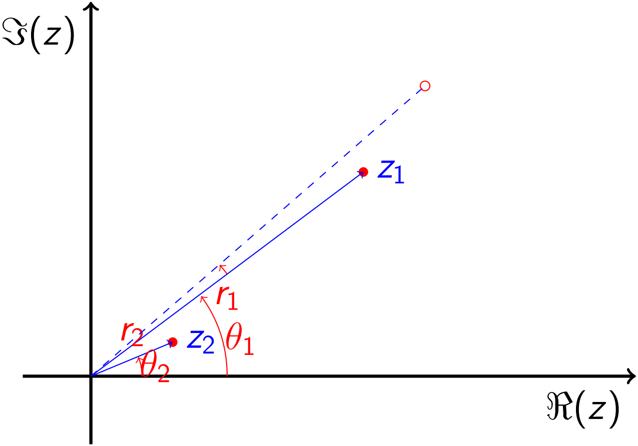

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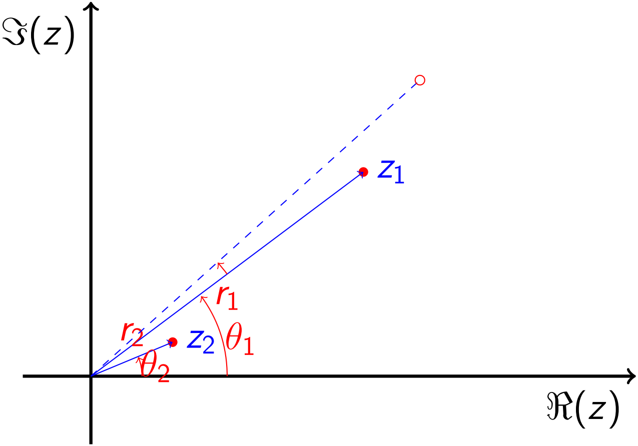

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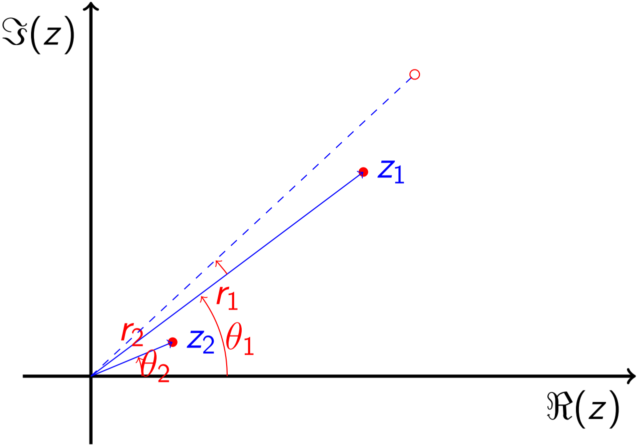

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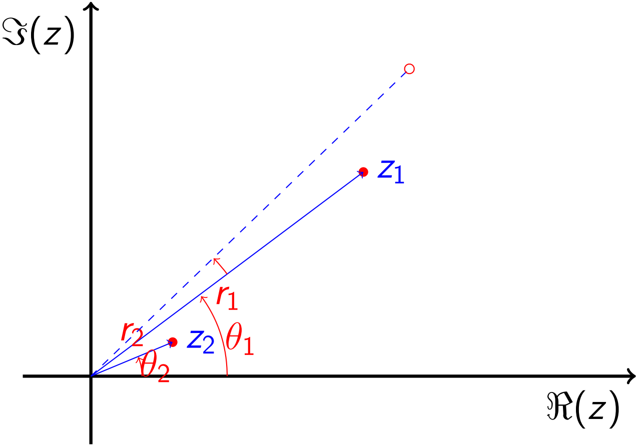

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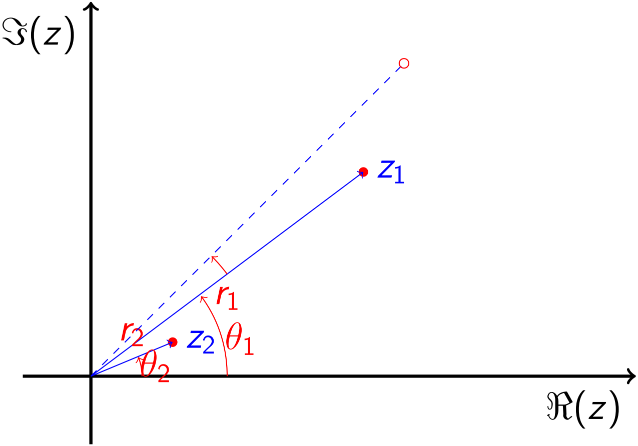

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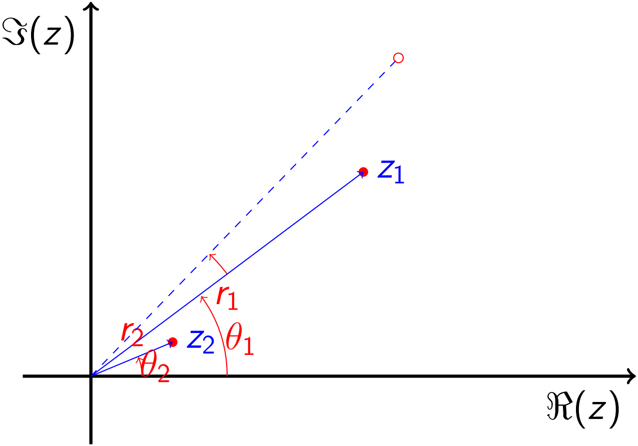

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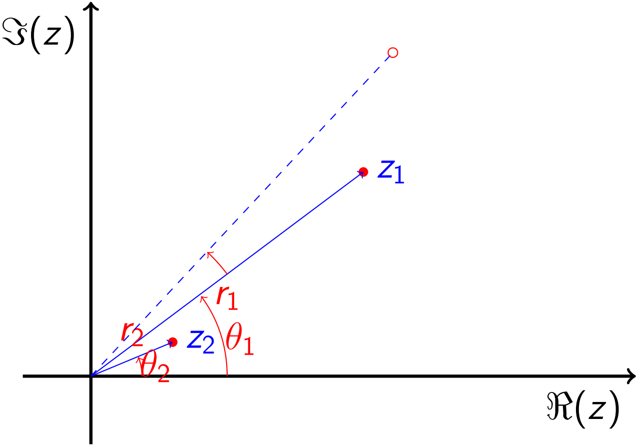

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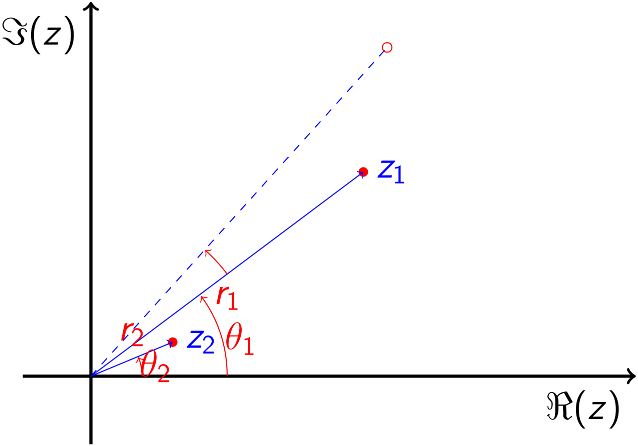

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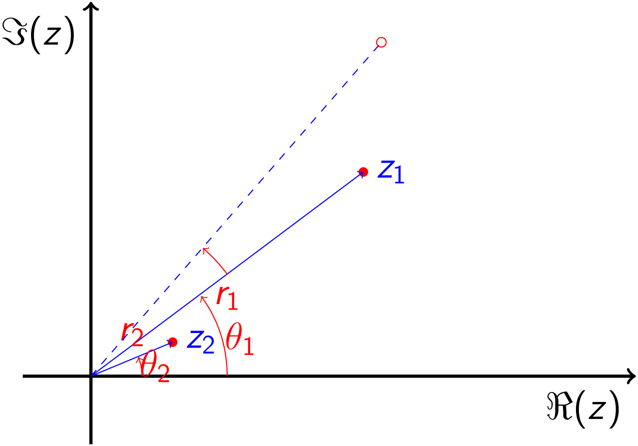

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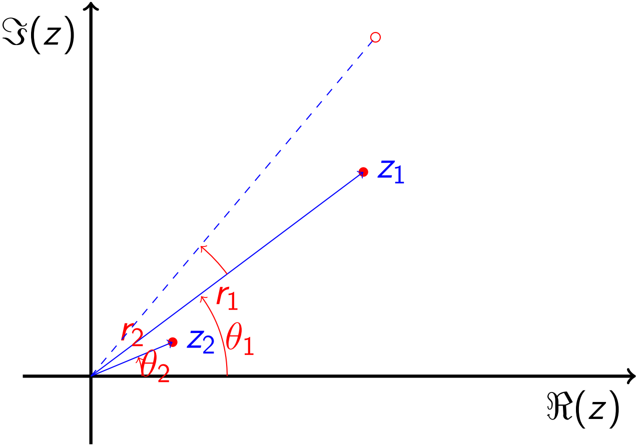

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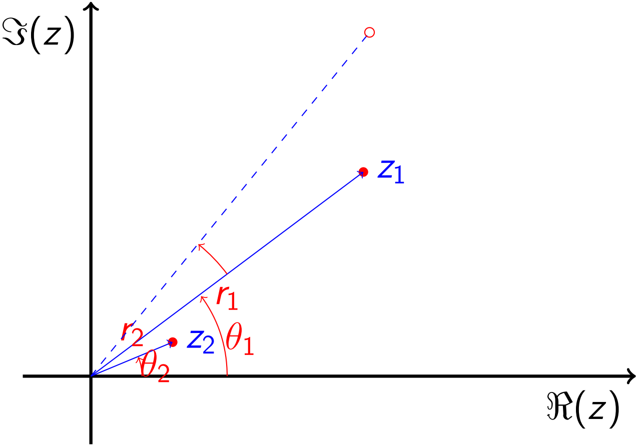

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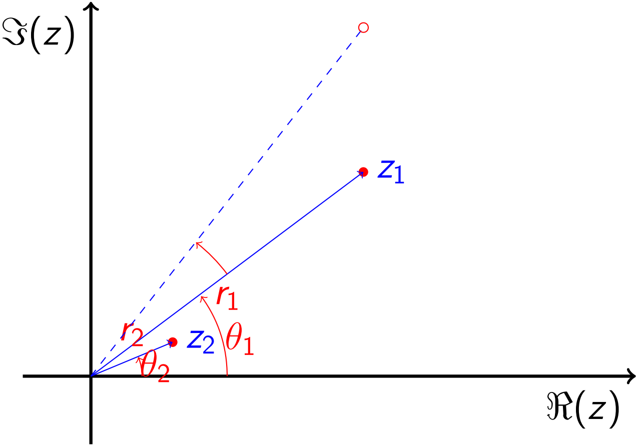

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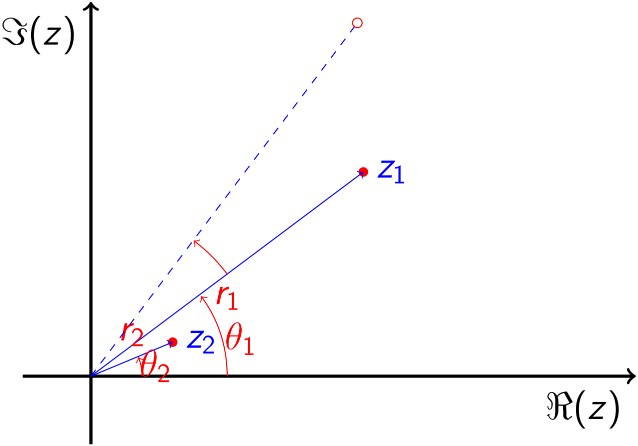

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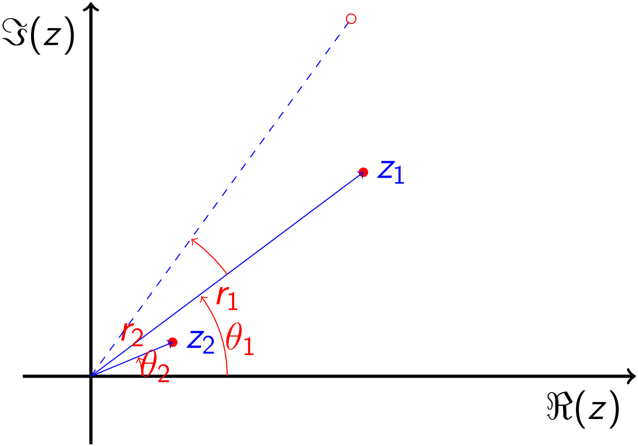

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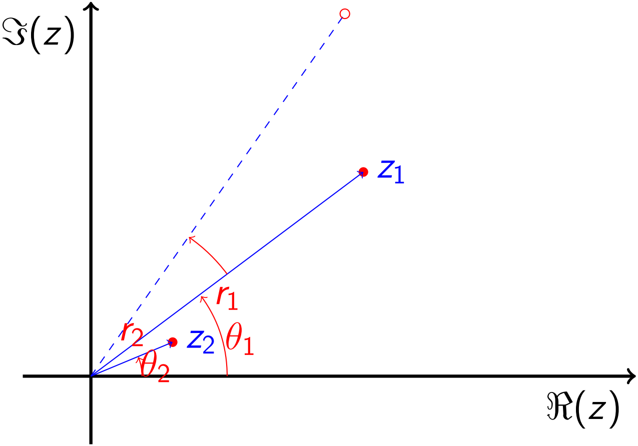

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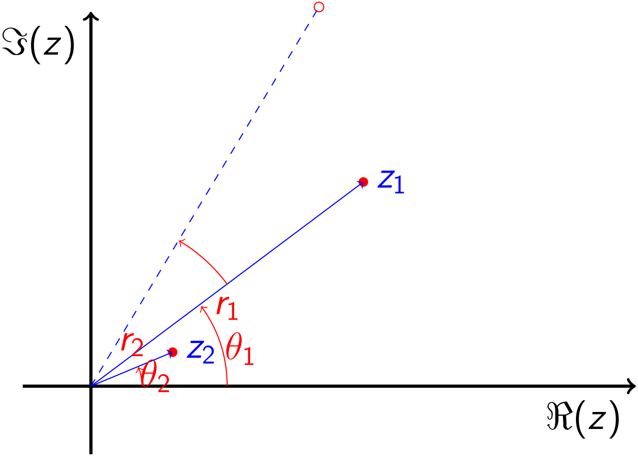

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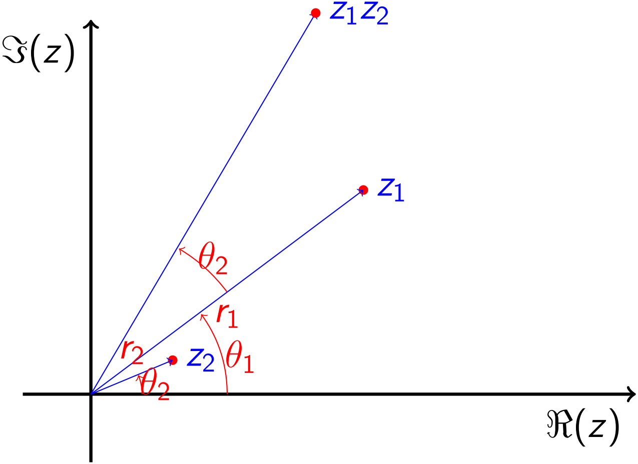

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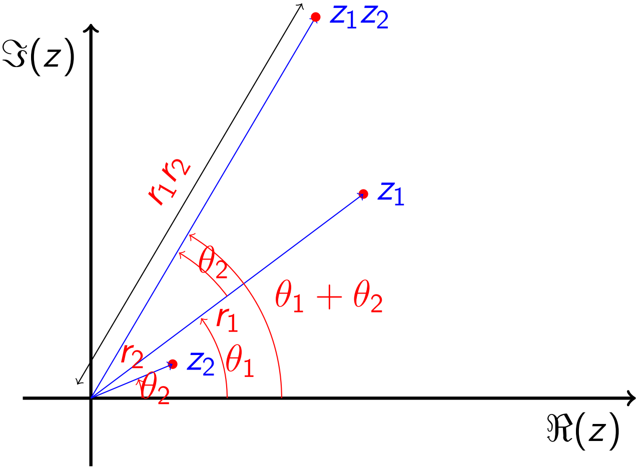

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Multiplying several complex numbers :

$z_1 z_2 \dots z_n$

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$$z^n = r^n \angle n\theta$$

Geometry behind Euler's roots of unity

A reminder :

The roots of $z^n = 1$ *lie on the vertices of a regular n-sided polygon inscribed inside a unit circle.*

- *Euler*

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The vertices $z_m,\ m \in \{0, 1, 2, \ldots, n-1\}$ are at $z_m = 1\angle\left(\frac{2\pi m}{n}\right)$.

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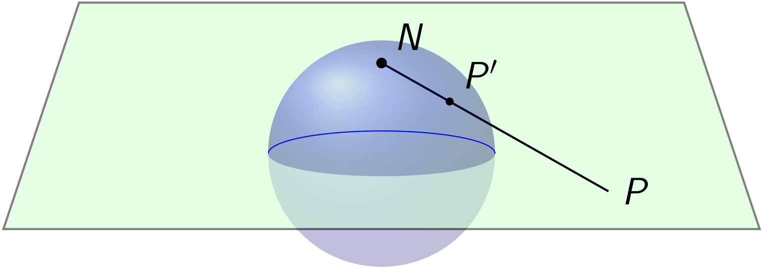
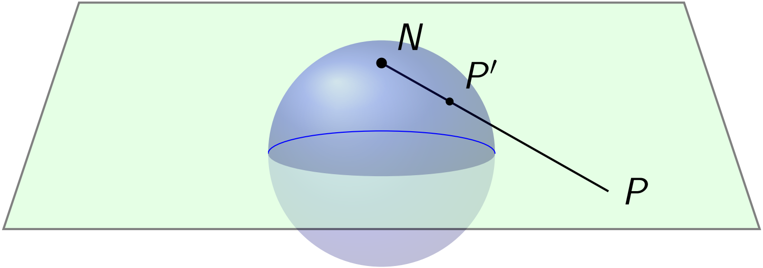
The Riemann sphere



The Riemann sphere is an alternative geometric representation of complex numbers.

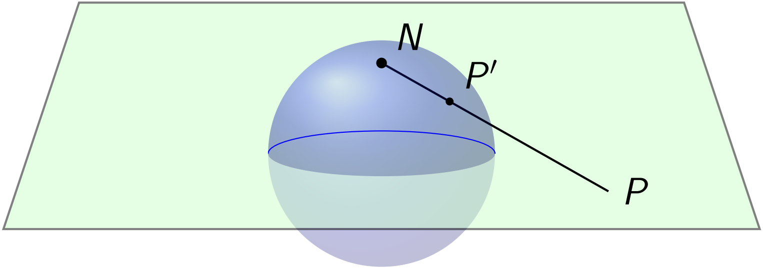
The Riemann sphere



The Riemann sphere is an alternative geometric representation of complex numbers.

The point P' where the straight line joining the "north pole" $N = (0, 0, 1)$ of the unit sphere centered at the origin to the point $P = (x, y, 0)$ represents the complex number $\zeta = x + iy$.

The Riemann sphere



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Numbers inside the unit circle, $|\zeta| < 1$, are represented by the southern hemisphere.

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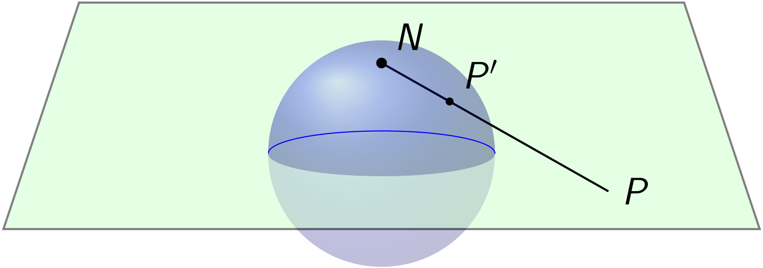

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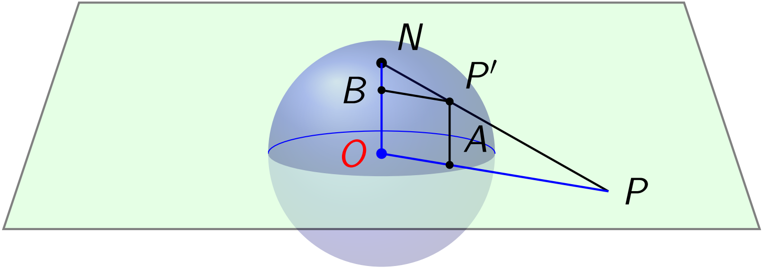
The north pole represents the *“point at infinity”!*

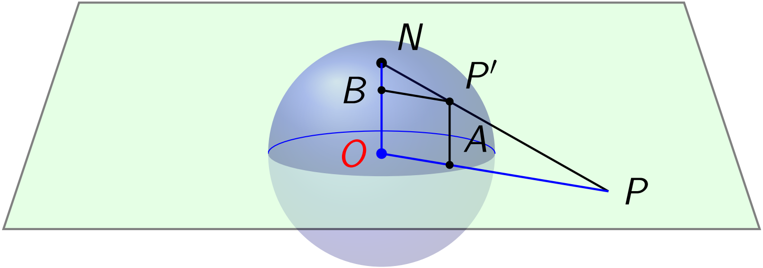
The Riemann sphere



$$O = (0,0,0), \quad N = (0,0,1), \quad P = (x,y,0)$$

$$P' = (x',y',z')\,, \quad A = (x',y',0) \quad B = (0,0,z')$$

The Riemann sphere



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The triangles $\Delta NPO, \Delta NP'B, \Delta P'PA$ are similar.

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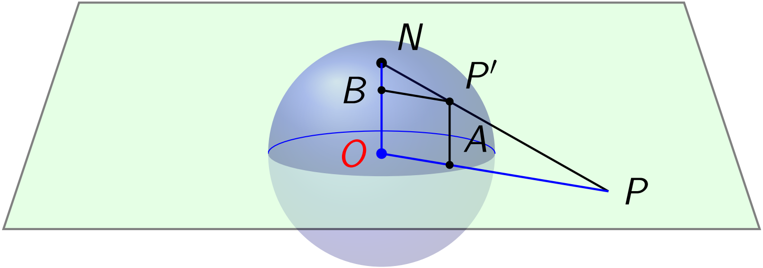

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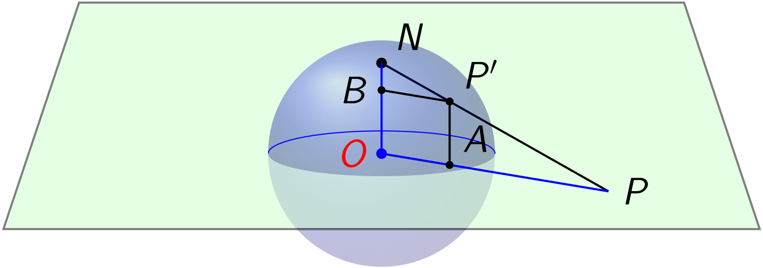
$$\frac{\overline{OA}}{\overline{OP}} = \frac{\overline{NP'}}{\overline{NP}} = \frac{\overline{NB}}{\overline{NO}} = 1 - z'$$

The Riemann sphere



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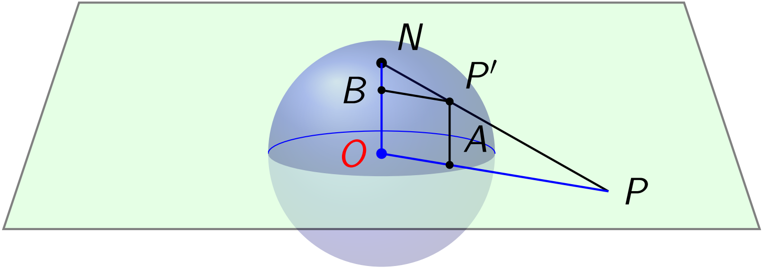
$$\frac{x'}{x} = \frac{y'}{y} = 1 - z'$$

The Riemann sphere



$$\frac{x'}{x} = \frac{y'}{y} = \frac{x' + \mathfrak{i}y'}{x + \mathfrak{i}y} = 1 - z'$$

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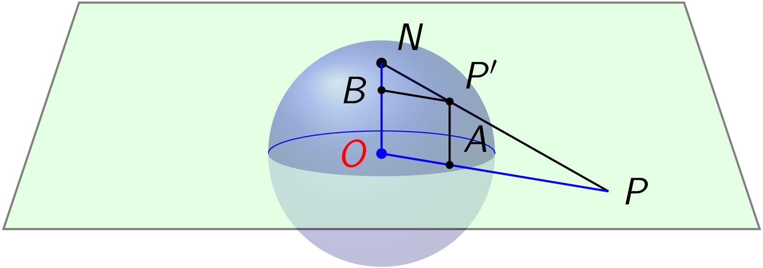

$$\frac{x'}{x} = \frac{y'}{y} = \frac{x' + iy'}{x + iy} = 1 - z'$$

$$\zeta = x + iy = \frac{x' + iy'}{1 - z'}$$

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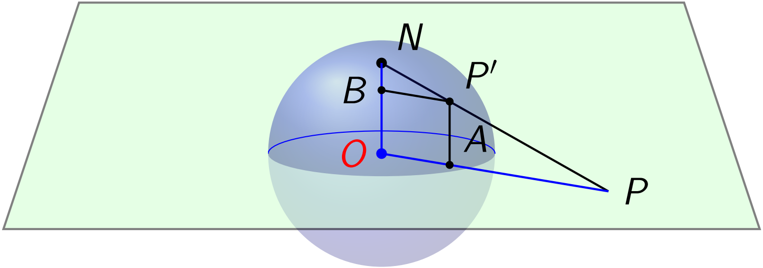

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$$|\zeta|^2 = \frac{x'^2 + y'^2}{(1 - z')^2}$$

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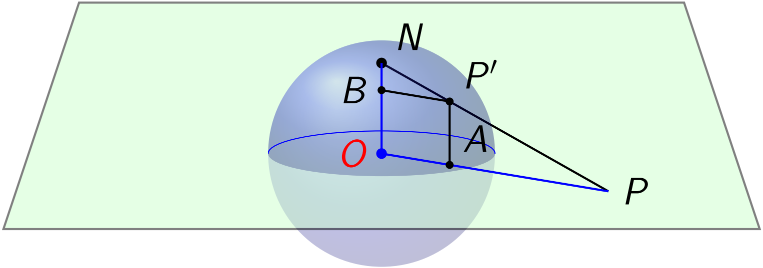
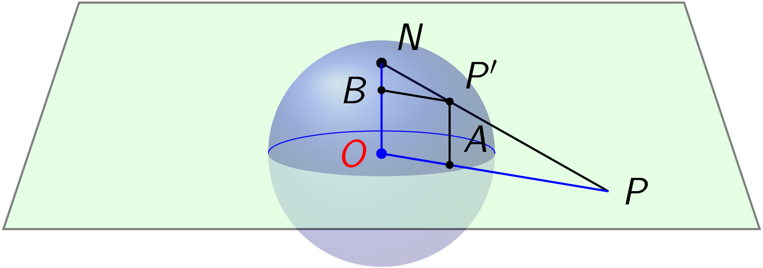


$$\frac{x'}{x} = \frac{y'}{y} = 1 - z'$$

$$\zeta = x + \mathrm{i}y = \frac{x' + \mathrm{i}y'}{1 - z'}$$

$$|\zeta|^2 = \frac{1 + z'}{1 - z'}$$

The Riemann sphere



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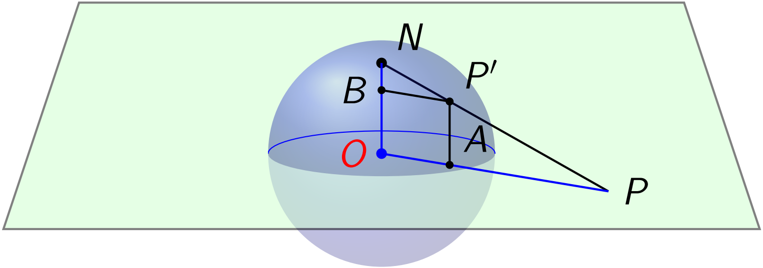

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$$x' = \frac{\zeta + \bar{\zeta}}{|\zeta|^2 + 1}, \qquad y' = -\mathrm{i}\,\frac{\zeta - \bar{\zeta}}{|\zeta|^2 + 1}, \qquad z' = \frac{|\zeta|^2 - 1}{|\zeta|^2 + 1}$$

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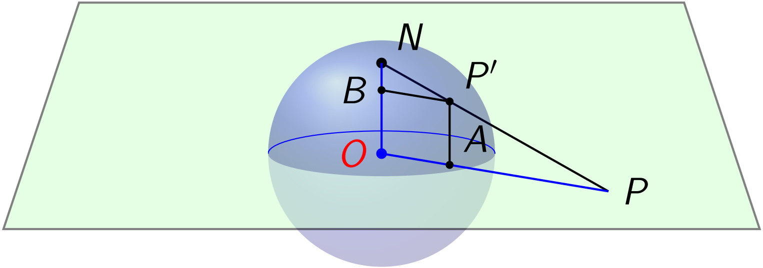

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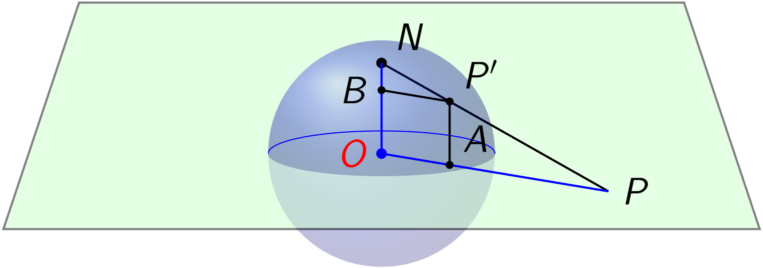

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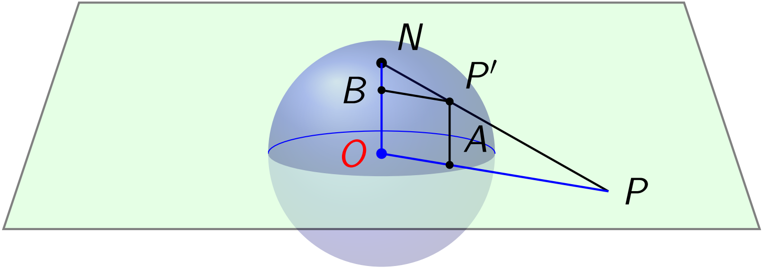

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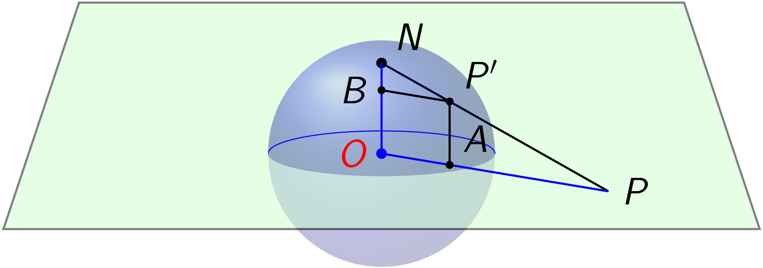

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- A finite sequence is an infinite sequence in which only a finite number of f_i are non-zero.
- Denote by $S(R)$ the set of all finite sequences of elements of R.

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Go Back!

The identity on C

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