Analyzing the Power Series method for linear ODEs contd.

Ananda Dasgupta

MA211, Lecture 20

Example - the hypergeometric DE

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There is one more regular singular point - at $z=\infty$!

Behaviour at infinity

The behaviour at ∞ of

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The point at ∞ is a regular singular point if

$$\frac{1}{t}P\left(\frac{1}{t}\right) \text{ and } \frac{1}{t^2}Q\left(\frac{1}{t}\right)$$

are analytic at $t = 0$.

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For the point at ∞ to be regular, we must have

$$p_1 = 2 \text{ and } q_2, q_3 = 0$$

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$z = \infty$ is a regular singular point of the hypergeometric DE

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The hypergeometric equation is one example. *Indeed, all SOHLODEs with three regular singular points can be recast in the form of the hypergeometric DE by a change of dependent and independent variables!*

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$$z^2 \sum_{n=0}^{\infty} (n+s)(n+s-1) c_n z^{n+s-2} + z \sum_{n=0}^{\infty} (n+s) c_n z^{n+s-1}$$

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The first two equations lead to :

$$s = \pm m \quad \text{and} \quad c_1 = 0$$

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The cofficients $c_2, c_4, \ldots, c_{2n}, \ldots$ can be written in terms of c_0 from the third.

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- However, when $m = 0$, we get only one solution!

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- Since s has two values, $\pm m$, we *expect* two independent solutions to the Bessel equation $J_m(z)$ and $J_{-m}(z)$.
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- One way is via the **Wronskian**.

The Wronskian

Given two solutions $f_1(z)$ and $f_2(z)$ of the SOLHODE

$$\frac{d^2f}{dz^2} + P(z)\frac{df}{dz} + Q(z)f(z) = 0$$

their Wronskian is defined to be

$$\Delta\left[f_1, f_2\right](z) = \begin{vmatrix} f_1(z) & f_2(z) \\ f_1'(z) & f_2'(z) \end{vmatrix}$$

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The converse is also true!

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Corrolary : Wronskian of two solutions is identically positive, identically negative, or identically zero.

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The Wronskian of two functions can be written as

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$$\begin{aligned} f_2(z) &= f_1(z) \int \frac{\Delta\left[f_1, f_2\right](z)}{\left(f_1(z)\right)^2} dz \\ &\propto f_1(z) \int \frac{\exp\left(-\int P(z)\, dz\right)}{\left(f_1(z)\right)^2} dz \end{aligned}$$

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- $f_2(z)$ has an essential singularity if P has a stronger pole.

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From the recursion relation, we get

$$J_0(z) = 1 - \frac{z^2}{4} + \frac{z^4}{64} - \frac{z^6}{2304} + \dots$$

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Transforming $z \to t = z^{-1}$

Back!

Define $g(t) = f\left(\frac{1}{t}\right)$.

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