Frobenius method for linear ODEs

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MA211, Lecture 21

The Frobenius method

If the point z_0 be a **regular singular point** of the SOHLODE

$$\frac{d^2 f}{dz^2} + P(z)\frac{df}{dz} + Q(z)f(z) = 0$$

we *expect* to find *two* independent solutions having the form

$$f(z) = \sum_{n=0}^{\infty} c_n \left(z - z_0\right)^{n+s}, \quad c_0 \neq 0$$

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Remember, z_0 is a regular singular point if both $(z - z_0) P(z)$ and $(z - z_0)^2 Q(z)$ are analytic at z_0 (*Strictly speaking, they may have removable singularities at* z_0).

Indicial and recursion relations

We choose $z_0 = 0$ to simplify the notation *without loss of generality*.

Indicial and recursion relations

$$z^2\frac{d^2f}{dz^2}+z\left(\pi_0+\pi_1 z+\ldots\right)\frac{df}{dz}+\left(\theta_0+\theta_1 z+\ldots\right)f=0$$

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$$\sum_{n=0}^{\infty} (n+s)(n+s-1) c_n z^{n+s}$$
$$+ \left(\pi_0 + \pi_1 z + \dots\right) \sum_{n=0}^{\infty} (n+s) c_n z^{n+s}$$
$$+ \left(\theta_0 + \theta_1 z + \dots\right) \sum_{n=0}^{\infty} c_n z^{n+s} = 0$$

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The indicial equation (from the coefficient of z^s):

$$F(s)\equiv s(s-1)+s\pi_0+\theta_0=0$$

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The coefficient of z^{s+1} :

$$[(s+1)sc_1+(s+1)\pi_0 c_1+\theta_0 c_1]+s\pi_1 c_0+\theta_1 c_0=0$$

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The coefficient of z^{s+2} :

$$F(s+2)c_2+s\pi_2 c_0+(s+1)\pi_1 c_1+\theta_2 c_0+\theta_1 c_1=0$$

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The coefficient of z^{s+n} :

$$F(s+n)c_n+\sum_{k=0}^{n-1}(s+k)\pi_{n-k}c_k+\sum_{k=0}^{n-1}\theta_{n-k}c_k=0$$

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For a given s, the coefficient c_n can be calculated from the recursion relation

$$F(s+n)c_n = -\sum_{k=0}^{n-1}(s+k)\pi_{n-k}c_k - \sum_{k=0}^{n-1}\theta_{n-k}c_k$$

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  - The series works for the bigger index.
  - For the smaller index $F(s+m) = 0$ and thus the method fails to yield c_m!
- In these cases the second solution can be found by other methods - *e.g.*the **Wronskian**.

Example - the Bessel equation

Consider the differential equation

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$$+ \sum_{n=0}^{\infty} c_n z^{n+s+2} - \sum_{n=0}^{\infty} m^2 c_n z^{n+s} = 0$$

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$$\left(s^2 - m^2\right) c_0 z^s + \left([s+1]^2 - m^2\right) c_1 z^{s+1}$$
$$+ \sum_{n=2}^{\infty} \left\{ \left[(n+s)^2 - m^2\right] c_n + c_{n-2} \right\} z^{n+s} = 0$$

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We can use this to calculate the coefficients $c_2, c_4, c_6, \ldots, c_{2m}, \ldots$ in terms of c_0.

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We can use this to calculate the coefficients $c_2, c_4, c_6, \ldots, c_{2m}, \ldots$ in terms of c_0. All odd order coefficients $c_1, c_3, \ldots$ are zero.

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  - For $m \in \mathbb{N}$ the two solutions are dependent.
  - For odd-half integer m the series gives a direct solution.
- Otherwise we get two solutions directly from the Frobenius method.

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For the root $s = +m$ of the indicial equation :

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$$c_n = \frac{-c_{n-2}}{(n+s)^2 - m^2} = -\frac{c_{n-2}}{n(n+2m)}$$

$$c_2 = -\frac{1}{2(2+2m)} c_0$$

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&= \frac{1}{2^4 1 \cdot 2(1+m)(2+m)} c_0 \\
c_6 &= -\frac{1}{2^6 1 \cdot 2 \cdot 3(1+m)(2+m)(3+m)} c_0
\end{aligned}$$

Example - the Bessel equation

The first solution

The general coefficient :

$$c_{2n} = \frac{(-1)^n}{2^{2n} 1 \cdot 2 \cdot 3 \cdots n(1+m)(2+m)\dots(n+m)} c_0$$

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Why?

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Why?

The first solution is

$$y_1(z) = c_0 z^m \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(m+1) z^{2n}}{2^{2n} n! \Gamma(n+m+1)}$$

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$$y_1(z) \;=\; c_0 \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(m+1) z^{2n+m}}{2^{2n} n! \Gamma(n+m+1)}$$

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y_1(z) &= c_0 \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(m+1) z^{2n+m}}{2^{2n} n! \Gamma(n+m+1)} \\
&= c_0 2^m \Gamma(m+1) \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n+m+1)} \left(\frac{z}{2}\right)^{2n+m}
\end{aligned}
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$$y_1(z) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(m+1) z^{2n+m}}{2^{2n} n! \Gamma(n+m+1)}$$
$$= c_0 2^m \Gamma(m+1) J_m(z)$$

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- For $m \in \mathbb{N}$ the solution y_2 diverges.
- In this case we can get a solution by choosing $c_0' 2^{-m}\Gamma(1-m) = d_0$ so that d_0 is finite:

$$y_2(z) = d_0\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n-m+1)}\left(\frac{z}{2}\right)^{2n-m}$$

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- This, though, is not an independent solution!

Example - the Bessel equation

The Bessel functions $J_{\pm m}(z)$

The two Bessel functions are defined as

$$J_{\pm m}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+1\pm m)} \left(\frac{z}{2}\right)^{2n\pm m}$$

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$$y(z) = AJ_m(z) + BJ_{-m}(z)$$

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The Bessel functions $J_{\pm m}(z)$

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$$J_{\pm m}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+1\pm m)} \left(\frac{z}{2}\right)^{2n\pm m}$$

For $m \notin \mathbb{Z}$, $J_{\pm m}(z)$ are independent and the general solution of the Bessel equation

$$y(z) = AJ_m(z) + BJ_{-m}(z)$$

If $m \in \mathbb{N}$:

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so this this does not give us the general solution.

Why?

Example - the Bessel equation with $m = \frac{1}{3}$

The solution for $s = +\frac{1}{3}$

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$$\phi_2(z) = c_0 z^{-\frac{1}{3}}\left(1 - \frac{3}{8}z^2 + \frac{9}{320}z^4 - \frac{9}{10240}z^6 + \ldots\right)$$

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&= Az^{\frac{1}{3}}\left(1 - \frac{3}{16}z^2 + \frac{9}{896}z^4 - \frac{9}{35840}z^6 + \dots\right) \\
&\quad + Bz^{-\frac{1}{3}}\left(1 - \frac{3}{8}z^2 + \frac{9}{320}z^4 - \frac{9}{10240}z^6 + \dots\right)
\end{aligned}
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Example - the Bessel equation with $m = 0$

The indicial equation in this case is

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All odd order coefficients are zero.

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&\vdots \\
c_{2m} &= \frac{(-1)^m}{2^m(m!)^2}c_0 \\
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\end{aligned}$$

Example - the Bessel equation with $m = 0$

One solution is

$$\phi_1(z) = c_0 \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{2^m (m!)^2}$$

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How can we find the second independent solution, $K_0(z)$?

Second solution via the Wronskian

The Wronskian of two functions can be written as

$$\Delta\left[f_1, f_2\right](z) = f_1(z)f_2'(z) - f_1'(z)f_2(z)$$

Second solution via the Wronskian

The Wronskian of two functions can be written as

$$\Delta\left[f_1, f_2\right](z) = \left(f_1(z)\right)^2 \frac{d}{dz}\left(\frac{f_2(z)}{f_1(z)}\right)$$

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$$f_2(z) = f_1(z) \int \frac{\Delta\left[f_1, f_2\right](z)}{\left(f_1(z)\right)^2} dz$$

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Thus, given a solution $f_1(z)$ to $f'' + Pf' + Qf = 0$, we can find another solution by

$$\begin{aligned} f_2(z) &= f_1(z) \int \frac{\Delta\left[f_1, f_2\right](z)}{\left(f_1(z)\right)^2} dz \\ &\propto f_1(z) \int \frac{\exp\left(-\int P(z)\, dz\right)}{\left(f_1(z)\right)^2} dz \end{aligned}$$

Back to Bessel (for $m = 0$)

$$z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} + z^2 f(z) = 0$$

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$$P(z) = \frac{1}{z}$$

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Remember that substituting $f(z) = \sum_{n=0}^{\infty} c_n z^{n+s}$ in the Bessel equation for $m = 0$

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Same solution as the one found before!

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The two Bessel functions $J_{\pm\frac{1}{2}}(z)$ are independent.

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&= \sum_{n=0}^{\infty} \frac{\sqrt{2}(-1)^n z^{2n+\frac{1}{2}}}{\sqrt{\pi}[2\cdot 4\cdots 2n][3\cdot 5\cdots (2n+1)]}
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The general solution is

$$\frac{A\sin(z) + B\cos(z)}{\sqrt{z}}$$

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$$\left(s^2 - 4\right) c_0 z^s + \left[(s+1)^2 - 4\right] c_1 z^{s+1}$$
$$+ \sum_{n=2}^{\infty} \left\{ \left[(n+s)^2 - 4\right] c_n + c_{n-2} \right\} z^{n+s} = 0$$

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Here the roots of the indicial equation differ by an integer.

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Let us ignore this inconsistency for the time being and prese ahead!

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We do get a solution by pressing ahead - but not a new solution!

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- This makes the right side $(s-2)^2(s+2)d_0 z^s$.
- $\left(\frac{\partial \tilde{f}}{\partial s}\right)_{s=2}$ is a solution!

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$$n! = \Gamma(1+n)$$

Dependence of $J_{\pm m}(z)$ for $m \in \mathbb{N}$

Go Back!

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&= (-1)^m J_m(z)
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