Complex integration

Ananda Dasgupta

MA211, Lecture 22

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$$F(t) = U(t) + \mathrm{i} V(t) \implies F'(t) = U'(t) + \mathrm{i} V'(t) :$$

$$\int_a^b f(t)dt = F(b) - F(a)$$

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- $\int_a^b (f(t) + g(t))\, dt = \int_a^b f(t)dt + \int_a^b g(t)dt$
- Integral triangle inequality :

$$\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|dt$$

The integral as a limit

Remember that the real integral $\int_a^b f(x)dx$ can be expressed as the limit as $\max_j |x_j - x_{j-1}| \to 0$ of the Riemann sum

$$\sum_{j=1}^{n} f(\tilde{x}_j)(x_j - x_{j-1})$$

where $x_j, \; j = 0, 1, \ldots, n$ form a partition of $[a, b]$:

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$$

and $\tilde{x}_j \in [x_{j-1}, x_j]$ are arbitrary.

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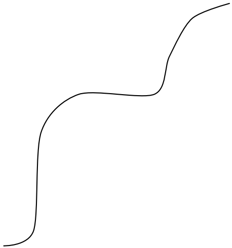
- The complex integral over a contour is defined in a similar way!
- Except that while there is only one way to go from a to b on the real line,
- there are infinite number of contours that can take you from z_1 to z_2 on the complex plane.

The integral as a limit



To define the **contour integral** of $f : \mathbb{C} \to \mathbb{C}$ over a contour C :

$$\int_C f(z)dz$$

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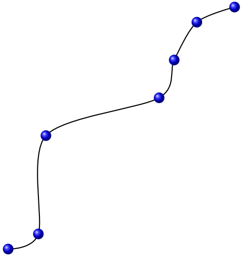

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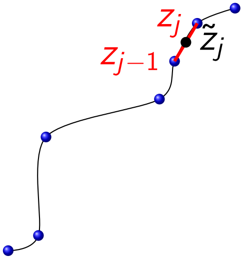

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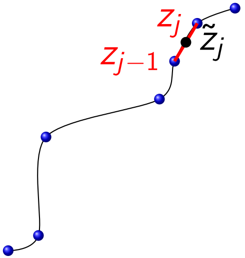

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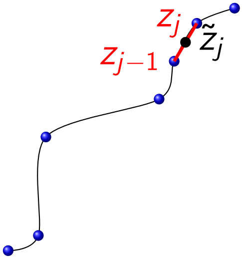

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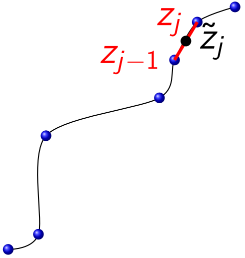

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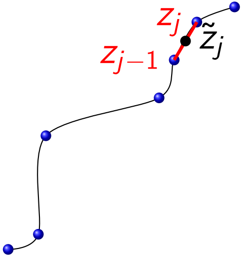

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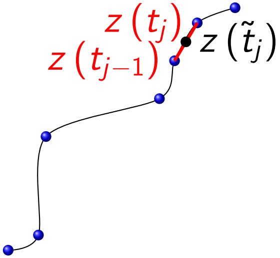

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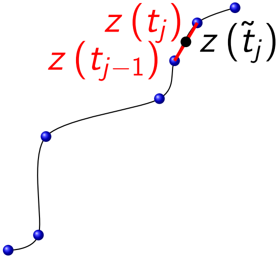

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- The j^{th} interval has endpoints $z(t_{j-1})$ and $z(t_j)$ and an arbitrary point in this interval is $z(\tilde{t}_j)$.

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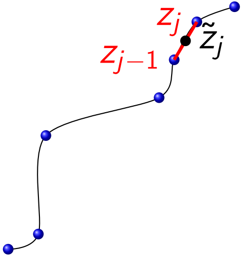

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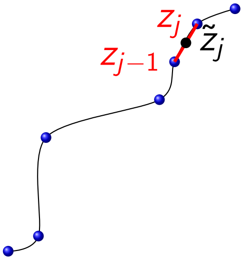

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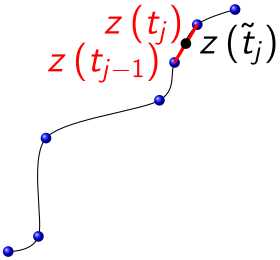

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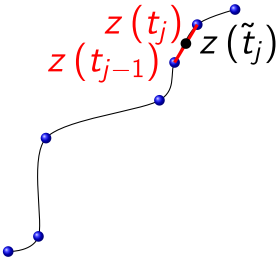

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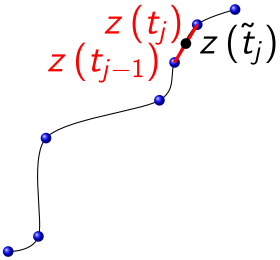

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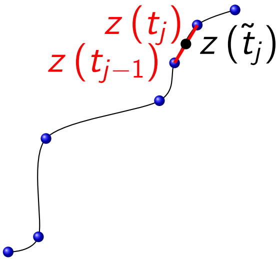

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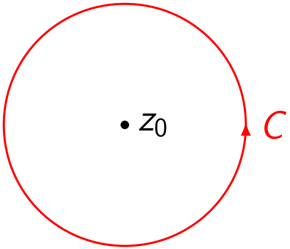
$$\int_C f(z)dz = \int_a^b f\left(z(t)\right) z'(t)dt$$

The value of the integral is indpendent of the actual parameterization chosen.

Examples

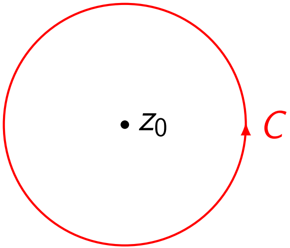


Consider the integral $\oint_C f(z)dz$ where C is the positively oriented circle

$$z(\theta) = z_0 + Re^{i\theta}, \; 0 \leq \theta < 2\pi$$

of radius R centered at z_0.

Examples



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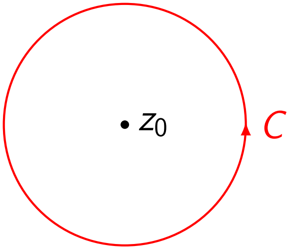

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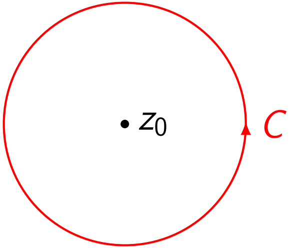
Our integral is

$$iR \int_0^{2\pi} f\left(z_0 + Re^{i\theta}\right) e^{i\theta} d\theta$$

Examples

Consider the special example

$$\oint_C (z - z_0)^n \, dz$$



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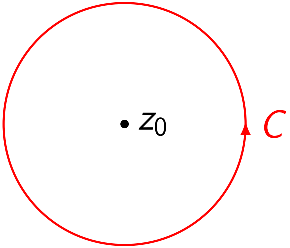

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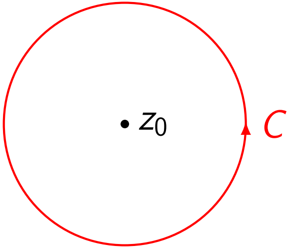

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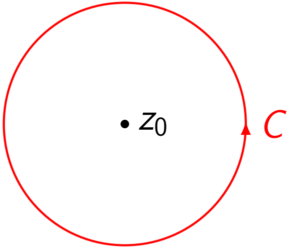

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for $n \neq -1$.

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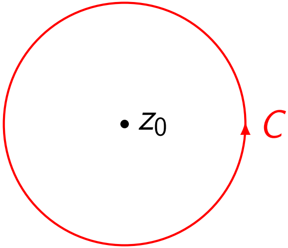

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$$= \mathrm{i}R^{n+1} \int_0^{2\pi} e^{\mathrm{i}(n+1)\theta} d\theta$$

$$= \mathrm{i}R^{n+1} \left. \frac{e^{i(n+1)\theta}}{\mathrm{i}(n+1)} \right|_0^{2\pi}$$

$$= \frac{R^{n+1}}{n+1} \left(e^{\mathrm{i}2\pi(n+1)} - 1\right)$$

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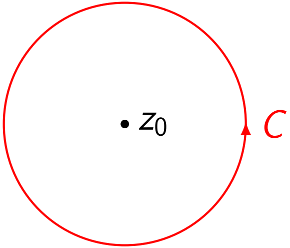

$$
\begin{aligned}
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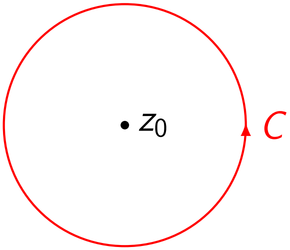
&= iR^{n+1} \left. \frac{e^{i(n+1)\theta}}{i(n+1)} \right|_0^{2\pi} \\
&= \frac{R^{n+1}}{n+1} \left(e^{i2\pi(n+1)} - 1\right) = 0
\end{aligned}
$$

for $n \neq -1$.

Examples

In the special case $n = -1$



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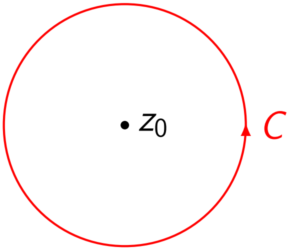

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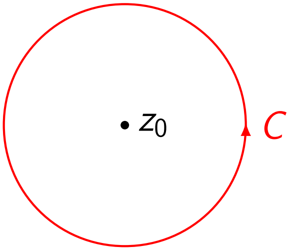

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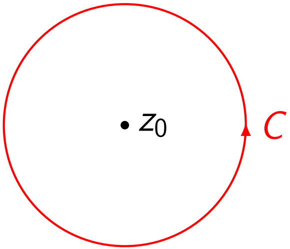

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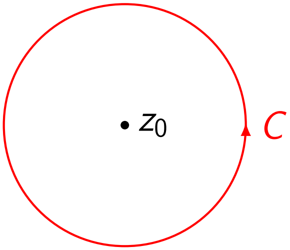
$$\oint_C \frac{dz}{z - z_0} = \int_0^{2\pi} \frac{\mathrm{i} R e^{\mathrm{i}\theta} d\theta}{R e^{\mathrm{i}\theta}} = \mathrm{i} \int_0^{2\pi} d\theta$$

Examples



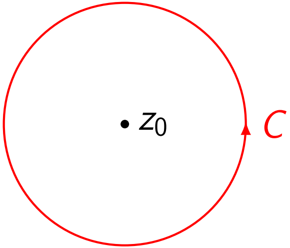
In the special case $n = -1$

$$\oint_C \frac{dz}{z - z_0} = \int_0^{2\pi} \frac{\mathrm{i} R e^{\mathrm{i}\theta} d\theta}{R e^{\mathrm{i}\theta}} = \mathrm{i} \int_0^{2\pi} d\theta = 2\pi \mathrm{i}$$

Examples

If $f(z)$ has the Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$



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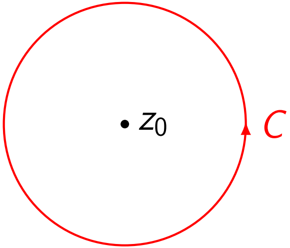

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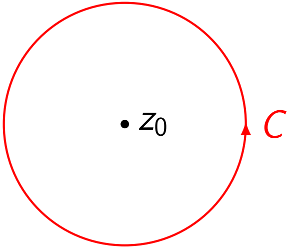

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$$\oint_C f(z)dz = \oint_C \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \, dz$$

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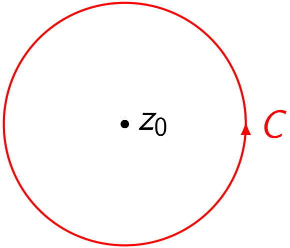

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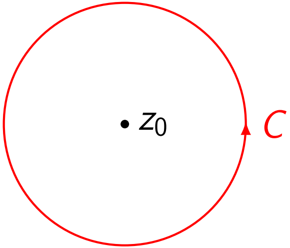

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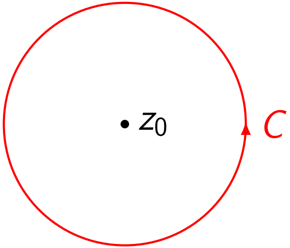

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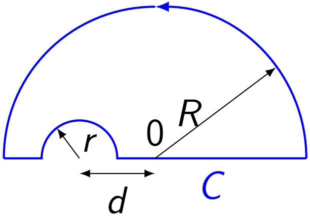
The coefficient a_{-1} is called the **residue** at z_0 of f.

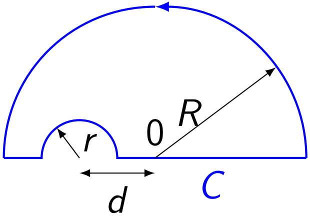
Examples



Consider

$$\oint_C f(z)dz$$

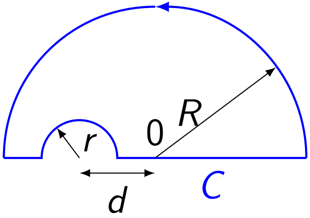
Examples



Consider

$$\oint_C f(z)dz = \oint_{\sum_{i=1}^4 C_i} f(z)dz$$

Examples



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$$\oint_C f(z)dz = \sum_{i=1}^{4} \int_{C_i} f(z)dz$$

Examples

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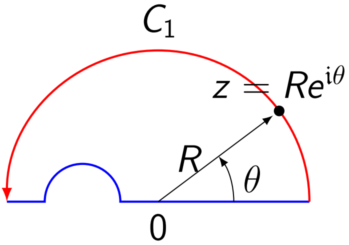
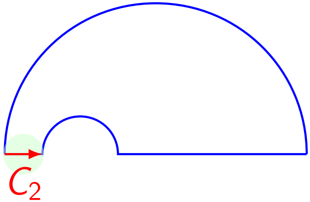
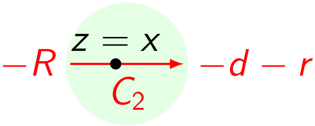


$$\int_{C_1} f(z)dz = \int_0^{\pi} f\left(Re^{i\theta}\right) iRe^{i\theta} d\theta$$

Examples





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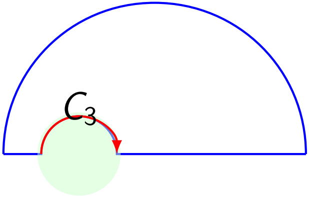
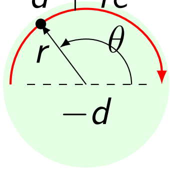
Examples





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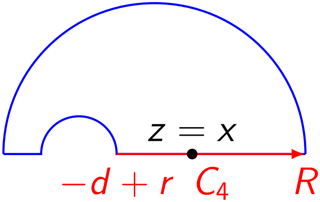

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