Complex integration
Cauchy's theorem

Ananda Dasgupta

MA211, Lecture 23

Properties of contour integrals

- $\int_{-C} f(z)dz = -\int_{C} f(z)dz$

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- $\displaystyle\int_{C_1+C_2} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$
- **The ML inequality :** If $f(z)$ is continuous on the contour Γ then

$$\left|\int_C f(z)dz\right| \leq ML$$

where M is an upperbound for the modulus $|f(z)|$ on C and L is the length of the contour C.

From contour integrals to real integrals

Let $f(z) = u(x, y) + \imath v(x, y)$, and

$$z(t) = x(t) + \imath y(t) \; a \leq t \leq b$$

is a paremeterization for a contour C. Then

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$$\begin{aligned}\int_C f(z)dz &= \int_a^b f(z(t))z'(t)dt \\ &= \int_a^b [u(x(t), y(t)) + \imath v(x(t), y(t))] \\ &\quad \times [x'(t) + \imath y'(t)]dt\end{aligned}$$

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Let $f(z) = u(x, y) + iv(x, y)$, and

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is a paremeterization for a contour C. Then

$$\begin{aligned}
\int_C f(z)dz &= \int_a^b f(z(t))z'(t)dt \\
&= \int_a^b \left[u\left(x(t), y(t)\right) + iv\left(x(t), y(t)\right)\right] \\
&\quad \times [x'(t) + iy'(t)]dt \\
&= \int_a^b \left[u(x, y)\, x'(t) - v(x, y)\, y'(t)\right] dt \\
&+ i \int_a^b \left[v(x, y)\, x'(t) + u(x, y)\, y'(t)\right] dt
\end{aligned}$$

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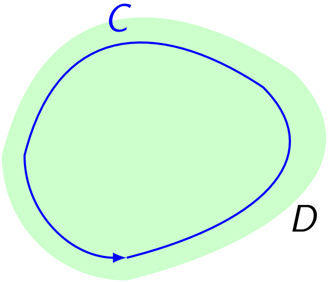
is a paremeterization for a contour C. Then

$$\int_C f(z)dz = \left[\int_C udx - vdy\right] + \imath \left[\int_C vdx + udy\right]$$

The Cauchy-Goursat theorem



Let f be holomorphic in a simply connected domain D. If C is a simple **closed** contour that lies in D, then

$$\boxed{\oint_C f(z)dz = 0}$$

Cauchy's proof of the Cauchy-Goursat theorem

The proof is simple if we make the additional assumption that $f'(z)$ is continuous.

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Theorem

Let C be a simple closed curve in $\mathbb{R}^2$ with positive orientation and let R be the interior of C. If M and N are continuous and have continuous partial derivatives $M_x, M_y, N_x,$ and N_y at all points on C and R, then

$$\oint_C M(x,y)dx + N(x,y)dy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dxdy$$

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If $f(z) = u(x, y) + iv(x, y)$ and $z(t) = x(t) + iy(t)$ is a parametrization of the contour C, we have

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If $f(z) = u(x, y) + \mathrm{i}v(x, y)$ and $z(t) = x(t) + iy(t)$ is a parametrization of the contour C, we have

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$$\begin{aligned}\oint_C f(z)dz &= \left[\oint_C udx - vdy\right] + \mathrm{i}\left[\oint_C udy + vdx\right] \\ &= \iint_R \left[-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] dxdy \\ &+ \mathrm{i}\iint_R \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] dxdy\end{aligned}$$

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&= \iint_R \left[-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] dxdy \\
&+ \mathrm{i} \iint_R \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] dxdy \\
&= -\iint_R (u_y + v_x)\, dxdy \\
&+ \mathrm{i} \iint_R (u_x - v_y)\, dxdy
\end{aligned}$$

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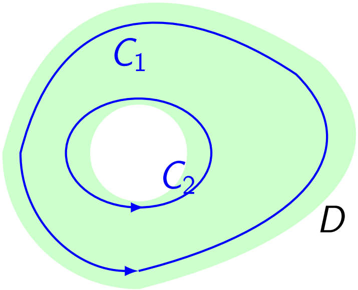
&= \iint_R \left[-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] dxdy \\
&+ \imath \iint_R \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] dxdy \\
&= -\iint_R (u_y + v_x)\, dxdy \\
&+ \imath \iint_R (u_x - v_y)\, dxdy = 0 \quad \text{(CR !)}
\end{aligned}$$

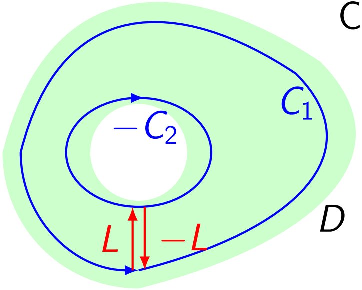
Deformation of contour



Let C_1 and C_2 be two simple closed positively oriented contours such that C_2 lies interior to C_1. If f is holomorphic in a domain D that contains both C_1 and C_2 and the region between them then

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$

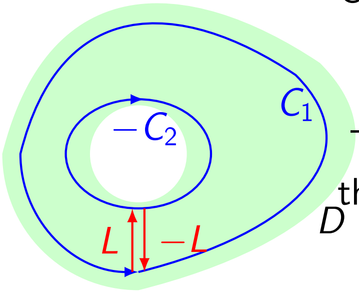
Proof



Consider the simple closed contour

$$C^* = C_1 + L + (-C_2) + (-L)$$

Proof

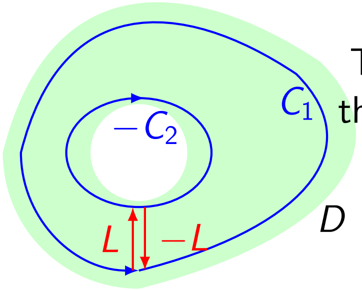


Consider the simple closed contour

$$C^* = C_1 + L + (-C_2) + (-L)$$

The function $f(z)$ is holomorphic in the interior of the loop C^*.

Proof



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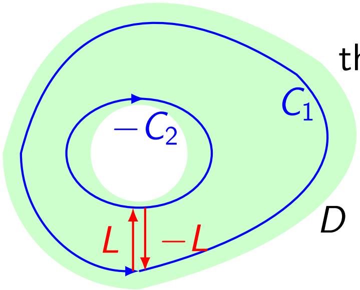

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$$\oint_{C_1+L+(-C_2)+(-L)} f(z)dz = 0$$

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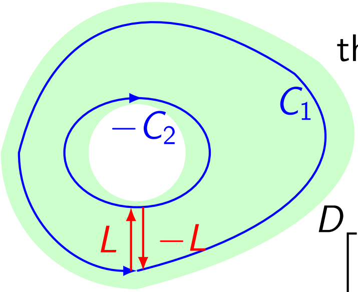
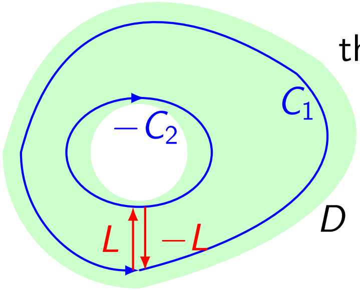


$$\left[\oint_{C_1} + \int_L + \oint_{-C_2} + \int_{-L}\right] f(z)dz = 0$$

Proof



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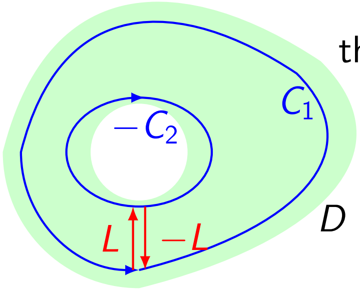

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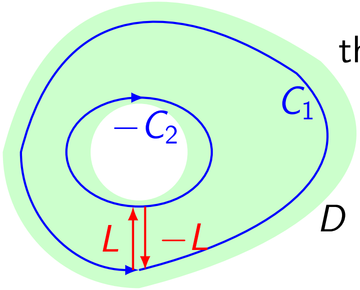

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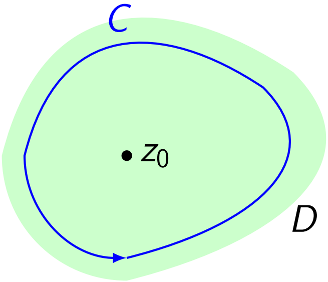
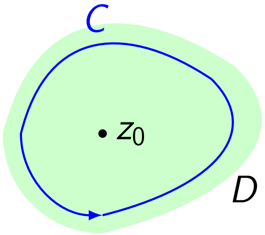
The Cauchy integral formula



Let f be holomorphic in the simply connected domain D and let C be a simple closed positively oriented contour that lies in D. If z_0 is a point that lies interior to C, then

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz$$

The Cauchy integral formula - Proof



f is holomorphic at $z_0 \implies$ that it is continuous there.

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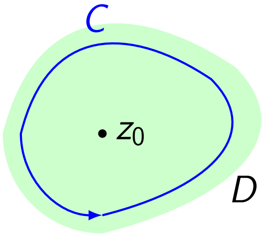
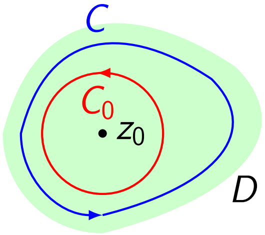


f is holomorphic at $z_0 \implies$ that it is continuous there. Thus given any $\epsilon > 0, \exists \delta$ such that

$$|z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon$$

The Cauchy integral formula - Proof



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Choose $0 < \alpha \leq 1$ such that $C_0 = \{|z - z_0| = \alpha\delta\}$ lies interior to C.

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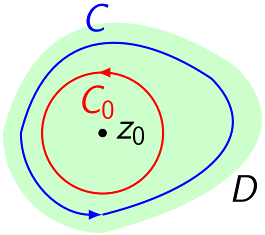

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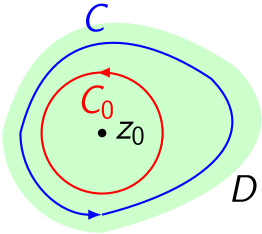

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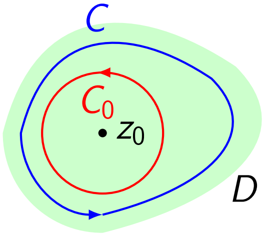

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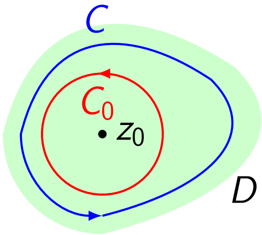

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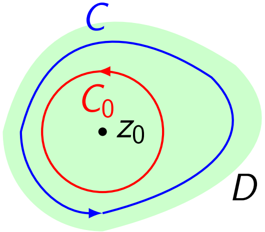
The deformation theorem says

$$\frac{1}{2\pi \mathrm{i}} \int_C \frac{f(z)\, dz}{z - z_0} = \frac{1}{2\pi \mathrm{i}} \int_{C_0} \frac{f(z)\, dz}{z - z_0}$$

The Cauchy integral formula - Proof



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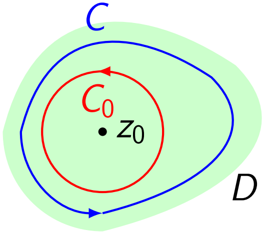

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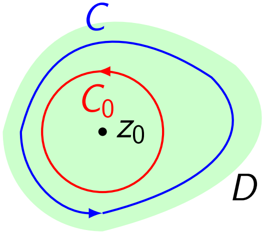
$$\left|\frac{1}{2\pi i}\int_{C} \frac{f(z)\,dz}{z - z_0} - f(z_0)\right| = \frac{1}{2\pi}\left|\int_{C_0} \frac{f(z) - f(z_0)}{z - z_0}dz\right|$$

The Cauchy integral formula - Proof



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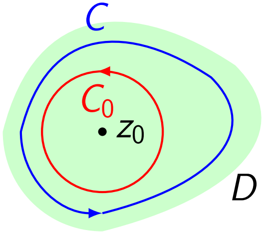
The Cauchy integral formula - Proof



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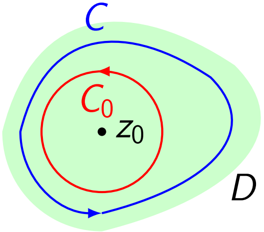
&\leq \frac{1}{2\pi}\int_{C_0} \frac{|f(z) - f(z_0)|}{|z - z_0|}|dz| \\
&\leq \frac{1}{2\pi}\frac{\epsilon}{\alpha\delta}2\pi\alpha\delta
\end{aligned}$$

The Cauchy integral formula - Proof



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$$\begin{aligned}
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&\le \frac{1}{2\pi}\frac{\epsilon}{\alpha\delta}2\pi\alpha\delta = \epsilon \qquad \square
\end{aligned}$$

Differentiation under the integral sign - Liebniz's rule

Let D be a simply connected domain, and let $I = [a, b] \subset \mathbb{R}$ be a closed interval.

Differentiation under the integral sign - Liebniz's rule

Let D be a simply connected domain, and let $I = [a, b] \subset \mathbb{R}$ be a closed interval. Let $f(z, t)$ and its partial derivative $f_z(z, t)$ with respect to z be continuous functions for all z in D and all t in I.

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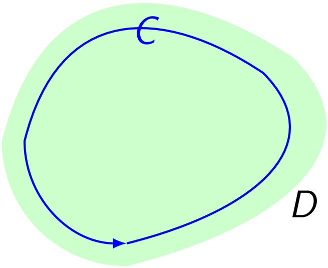
The Cauchy integral formula for derivatives



Let f be holomorphic in the simply connected domain D, and let C be a simple closed positively oriented contour that lies in D. If z is a point that lies interior to C, then

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz}$$

The Cauchy integral formula for derivatives - proof :

Parameterize the contour C by $z(t), a \leq t \leq b$.

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The Cauchy integral formula for derivatives - proof :

Parameterize the contour C by $z(t), a \leq t \leq b$.

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We can continue with the function $f'(z)$ and inductively prove the formula for all n.

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Let f be holomorphic in the simply connected domain D that contains the circle $C \ : \ |z - z_0| = R$.

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Taking $R \to \infty$ gives $|f'(z_0)| = 0$ for arbitrary z_0. $\square$

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If $P(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial of degree $n > 0$, then it has at least one zero.

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Liouville's theorem proves that $f(z)$ is a constant – *a contradiction*. $\square$

Goursat's proof of the Cauchy-Goursat theorem

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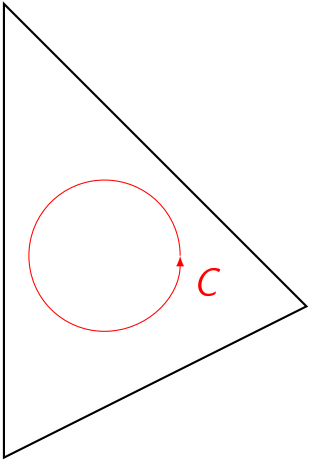
Goursat gave a proof of Cauchy's theorem that does not depend on this additional assumption.

Goursat's proof of the Cauchy-Goursat theorem

An outline :



We consider the contour integral

$$\oint_C f(z)dz$$

where C is a triangular contour which is contained in a simply connected domain where f is holomorphic.

We want to show that this vanishes.

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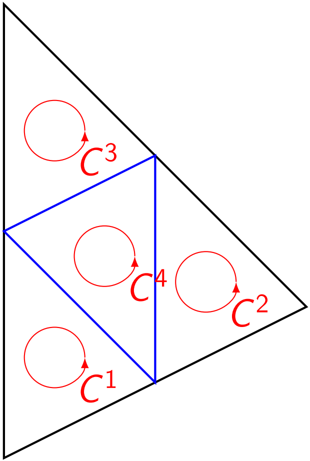

Split the triangle into four equal pieces.

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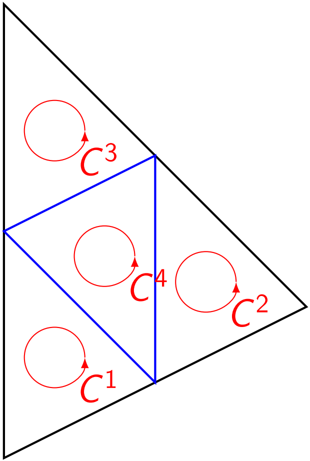

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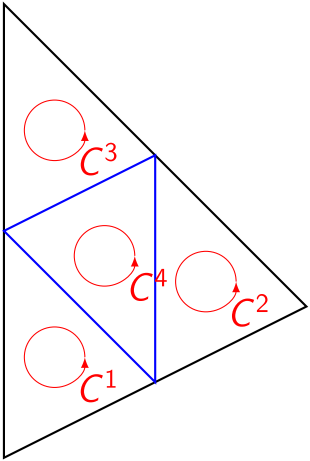

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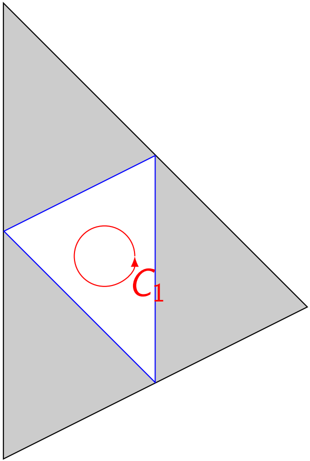

We choose that particular C^i as $C_1$and repeat the procedure.

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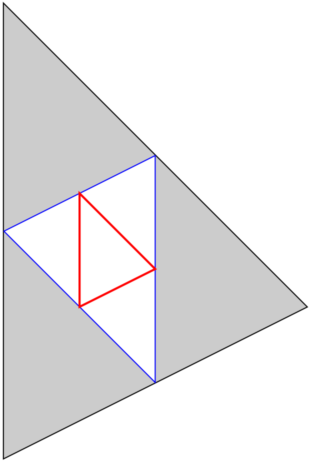

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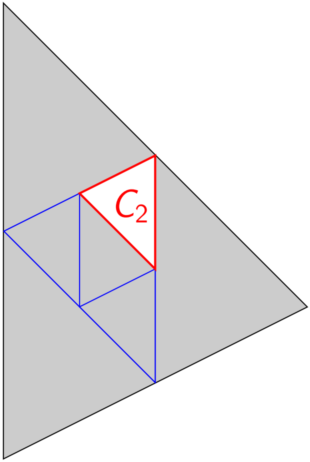

We choose that particular C^i as $C_1$and repeat the procedure. This gives us C_2.

Goursat's proof of the Cauchy-Goursat theorem

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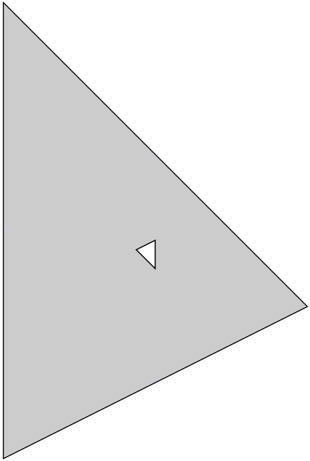

Repating this procedure gives us a sequence of triangular contours (C_n) such that

$$\text{interior of } C_{n+1} \subset \text{interior of } C_n$$

and

$$\left| \oint_{C_n} f(Z)dz \right| \leq 4 \left| \oint_{C_{n+1}} f(Z)dz \right|$$

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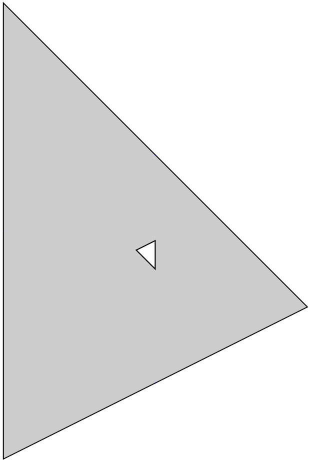

Our original integral is bounded by

$$\left|\oint_C f(Z)dz\right| \leq 4^n \left|\oint_{C_n} f(Z)dz\right|$$

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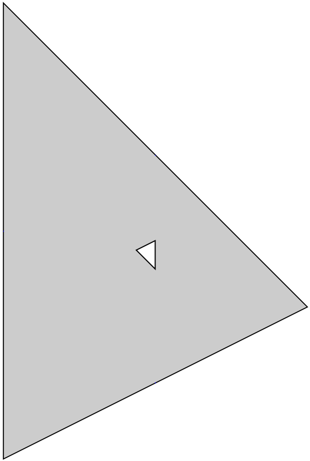

Let us denote the triangle C_n and its interior by T_n.

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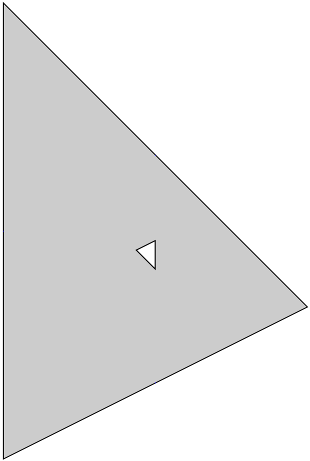

Let us denote the triangle C_n and its interior by T_n.

By Cantor's theorem $\exists z_0 \in \mathbb{C}$:

$$\bigcap_{n=1}^{\infty} T_n = \{z_0\}$$

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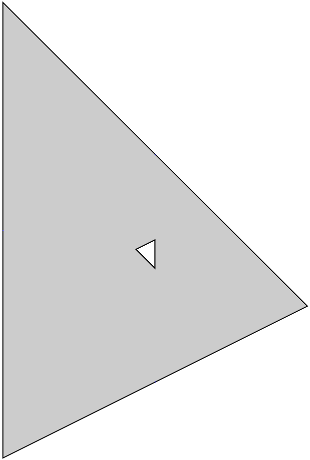

Since $f(Z)$ is holomorphic at z_0, $\exists \eta(z)$ such that

$$\begin{aligned} f(z) &= f(z_0) + f'(z_0)(z - z_0) \\ &+ \eta(z)(z - z_0) \end{aligned}$$

and

$$\lim_{z \to z_0} \eta(z) = 0$$

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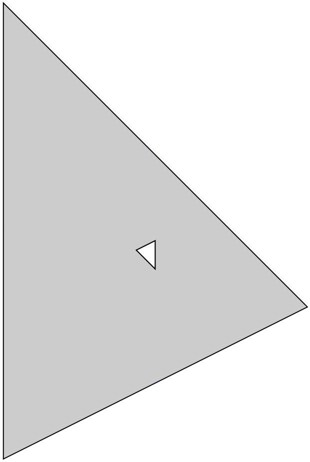

Now,

$$\begin{aligned}\oint_{C_n} f(z)dz &= \oint_{C_n} f(z_0)\,dz \\ &+ \oint_{C_n} f'(z_0)(z - z_0)\,dz \\ &+ \oint_{C_n} \eta(z)(z - z_0)\,dz\end{aligned}$$

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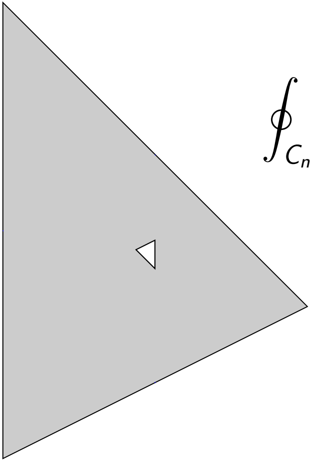

$$\begin{aligned}\oint_{C_n} f(z)dz &= \left[f(z_0) - f'(z_0)\, z_0\right] \oint_{C_n} dz \\ &+ f'(z_0) \oint_{C_n} z dz \\ &+ \oint_{C_n} \eta(z)\,(z - z_0)\, dz\end{aligned}$$

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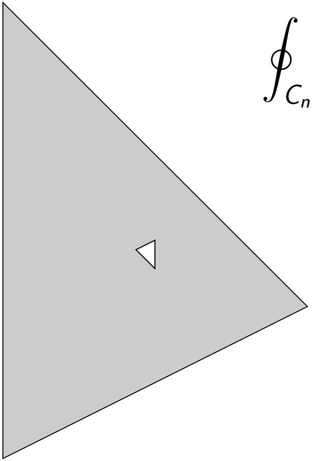

$$\begin{aligned}
\oint_{C_n} f(z)dz &= \left[f\left(z_0\right) - f'\left(z_0\right) z_0\right] \oint_{C_n} dz \\
&+ f'\left(z_0\right) \oint_{C_n} zdz \\
&+ \oint_{C_n} \eta(z)\left(z - z_0\right) dz
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$$\oint_{C_n} f(z)dz = \oint_{C_n} \eta(z)\left(z - z_0\right) dz$$

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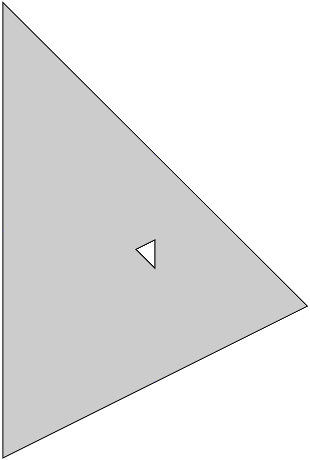

$\because \lim_{z \to z_0} \eta(z) = 0$, $\forall \epsilon > 0$
$\exists \delta > 0$:

$$|z - z_0| < \delta \implies |\eta(z)| < \frac{\epsilon}{L^2}$$

where L is the perimeter of the triangle C.

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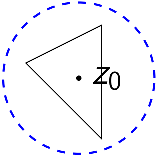
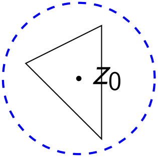


Choose a $n \in \mathbb{N}$ such that

$$T_n \subset B_\delta(z_0)$$

Goursat's proof of the Cauchy-Goursat theorem

An outline :



Choose a $n \in \mathbb{N}$ such that

$$T_n \subset B_\delta\left(z_0\right)$$

For all z on C_n we must have

$$|z - z_0| < \frac{L_n}{2}$$

where L_n is the perimeter of C_n.

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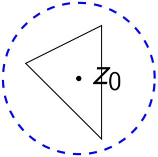

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$$|z - z_0| < \frac{L_n}{2} = \frac{L}{2^{n+1}}$$

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$$\left| \oint_C f(z)dz \right| \;\leq\; 4^n \left| \oint_{C_n} f(z)dz \right|$$

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We can carry the proof on to a general polygon by subdividing it into triangles, and onto a general closed contour by approximating it with a polynomial.