Complex integration
Applications of the Residue theorem

Ananda Dasgupta

MA211, Lecture 24

Residues

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The Residue theorem

Let D be a simply connected domain, and let C be a simple closed positively oriented contour that lies in D. If f is analytic inside C and on C, except at the points $z_1, z_2, \ldots, z_n$ that lie *inside* C, then

$$\int_C f(z)dz = 2\pi \mathrm{i} \sum_{k=1}^{n} \mathrm{Res}_{z=z_k} f$$

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Collecting terms it is easy to see that the coefficient of $\frac{1}{z}$ is zero!

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$$f(z) = \frac{a^{-1}}{z - z_0} + a_0 + a_1 (z - z_0) + \ldots$$

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and thus

$$\mathrm{Res}_{z=z_0} f(z) = \lim_{z \to z_0} [(z - z_0) f(z)]$$

Calculating residues

Higher order pole

If z_0 is a pole of f of order n it has a Laurent expansion

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \ldots + \frac{a^{-1}}{z - z_0} + a_0 + a_1 (z - z_0) + \ldots$$

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Hence a_{-1} is the coefficient of $(z - z_0)^{n-1}$ in the Taylor expansion of $(z - z_0)^n f(z)$ and thus

$$\mathrm{Res}_{z=z_0} f(z) = \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[(z - z_0)^n f(z) \right] \Big|_{z=z_0}$$

Fun with residues

A formula for Fibonacci numbers

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f_n is the residue at $z = 0$ of

$$\frac{F(z)}{z^{n+1}} = \frac{1}{z^{n+1}\left(1 - z - z^2\right)}$$

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$$f_n = -\sum_{z_p=z_+,z_-} \mathrm{Res}_{z=z_p} \frac{1}{z^{n+1}\left(1-z-z^2\right)}$$

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$$Res_{z=z_+} \frac{1}{z^{n+1}(1-z-z^2)} = -\lim_{z \to z_+} \frac{z-z_+}{z^{n+1}(z-z_+)(z -$$

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$$Res_{z=z_+} \frac{1}{z^{n+1}(1-z-z^2)} = -\frac{(-z_-)^{n+1}}{z_+ - z_-}$$

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$$Res_{z=z_+} \frac{1}{z^{n+1}(1-z-z^2)} = -\frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}+1}{2}\right)^{n+1}$$

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$$\boxed{f_n = \frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}+1}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}$$

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Calculation of trigonometric integrals

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- The integral becomes

$$-i \int_C f\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right) z^{-1} dz$$

where C is the positively oriented unit circle, $|z| = 1$.

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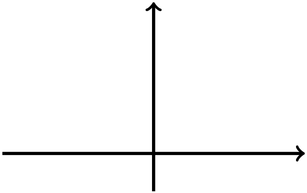

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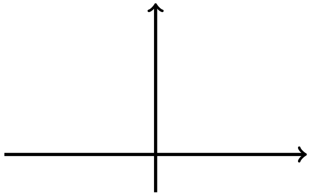

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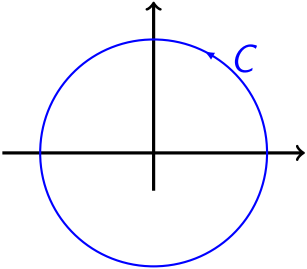

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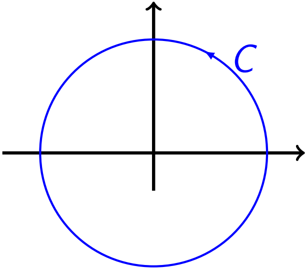

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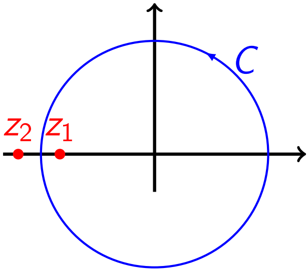

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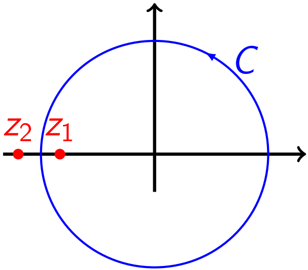

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the larger of which, z_1, is inside the unit circle.

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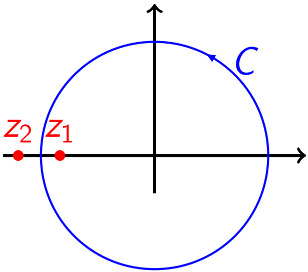

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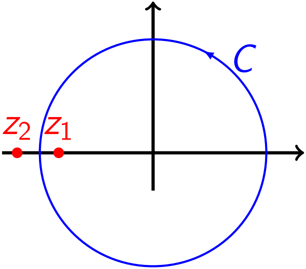

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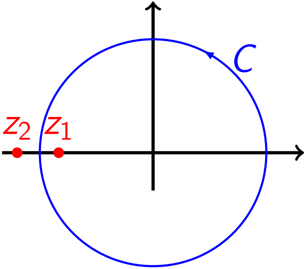

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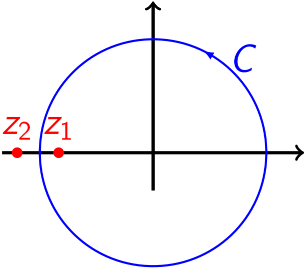

Write

$$z = e^{i\theta} \implies \cos(\theta) = \frac{1}{2}\left(z + \frac{1}{z}\right).$$

$$\int_0^{2\pi} \frac{d\theta}{1+\varepsilon\cos(\theta)} = \int_C \frac{-iz^{-1}dz}{1+\frac{\varepsilon}{2}\left(z+\frac{1}{z}\right)}$$

where C is the unit circle $|z| = 1$.

$$\mathrm{Res}_{z=z_1} \frac{-2i}{\varepsilon z^2 + 2z + \varepsilon} = \frac{-2i}{\varepsilon\left(z_1 - z_2\right)}$$

Applications of the residue theorem

Calculation of trigonometric integrals

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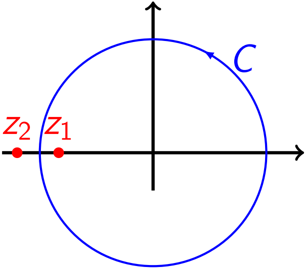

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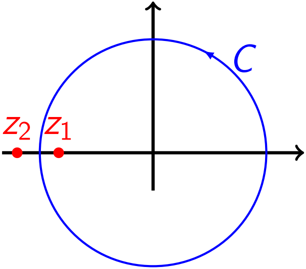

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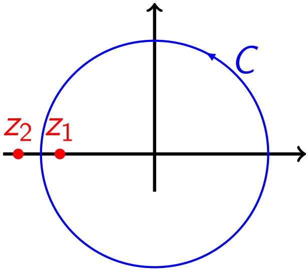

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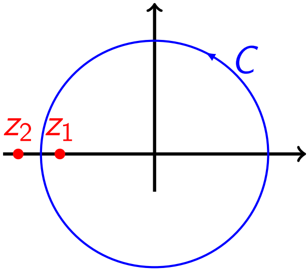

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Applications of the residue theorem

Calculation of trigonometric integrals

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The integrand

$$-i \frac{\left(z^2+1\right)^{2n}}{2^{2n} z^{2n+1}}$$

has a pole of order $2n+1$ at $z = 0$.

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Applications of the residue theorem

Calculation of trigonometric integrals

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$$-\frac{i}{2^{2n}} \binom{2n}{n}$$

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The integral is

$$\int_0^{2\pi} \cos^{2n}(\theta)d\theta = 2\pi \frac{(2n)!}{2^{2n}(n!)^2}$$

Consider a real integral of the form

$$I = \int_{-\infty}^{\infty} f(x)dx$$

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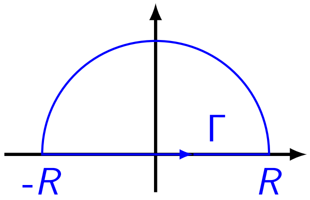
- $f(z) \to 0$ at least as fast as $\frac{1}{|z|^{\alpha}}$ as $|z| \to \infty$ in the upper half plane for $\alpha > 1$.

Applications of the residue theorem

Calculation of real integrals - case 1

Evaluate $\oint_{\Gamma} f(z)dz$

Evaluate $\oint_\Gamma f(z)dz$ along the contour Γ:



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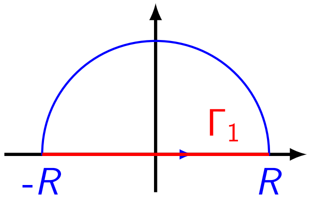

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Applications of the residue theorem

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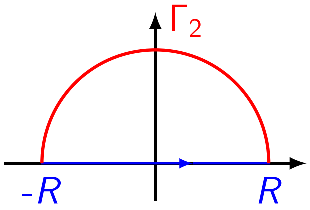

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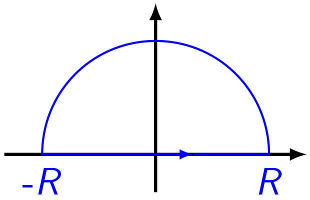

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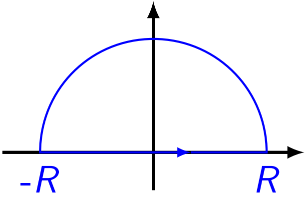

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Applications of the residue theorem

Calculation of real integrals - case 1

- $\int_{\Gamma_1} f(z)dz = \int_{-R}^{R} f(x)dx$, so that

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- The residue theorem says that this is $2\pi\mathrm{i}\times$ Sum of residues of $f(z)$ at its poles in the upper half plane.

Applications of the residue theorem

Calculation of real integrals

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$$

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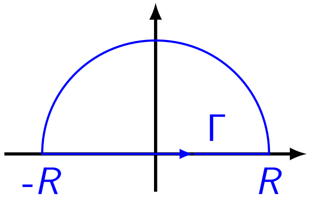

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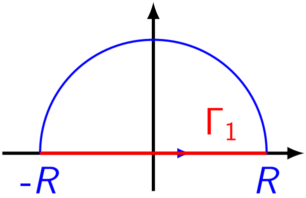

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Applications of the residue theorem

Calculation of real integrals

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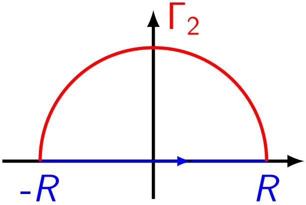

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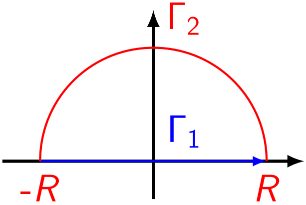
$$\int_{\Gamma_2} \frac{dz}{1+z^4} =$$

$$\int_0^{\pi} \frac{iRe^{i\theta} d\theta}{1+R^4 e^{i4\theta}}$$

Applications of the residue theorem

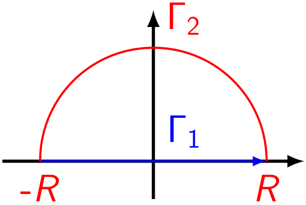
Calculation of real integrals



$$\int_{\Gamma} \frac{dz}{1+z^4} = \int_{\Gamma_1} \frac{dz}{1+z^4} + \int_{\Gamma_2} \frac{dz}{1+z^4}$$

Applications of the residue theorem

Calculation of real integrals

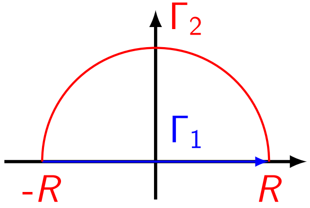


$$\int_\Gamma \frac{dz}{1+z^4} = \int_{\Gamma_1} \frac{dz}{1+z^4} + \int_{\Gamma_2} \frac{dz}{1+z^4}$$

$$\lim_{R\to\infty} \oint_\Gamma \frac{dz}{1+z^4} = \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{1+x^4} + \lim_{R\to\infty} \int_0^{\pi} \frac{iRe^{i\theta}d\theta}{1+R^4e^{i4\theta}}$$

Applications of the residue theorem

Calculation of real integrals



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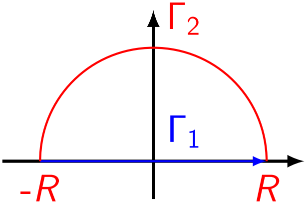
Applications of the residue theorem

Calculation of real integrals



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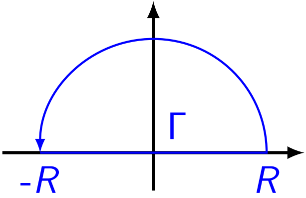
$\because \lim_{R\to\infty} I_2 = 0$ we have

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \lim_{R\to\infty} \oint_{\Gamma} \frac{dz}{1+z^4}$$

Applications of the residue theorem

Calculation of real integrals

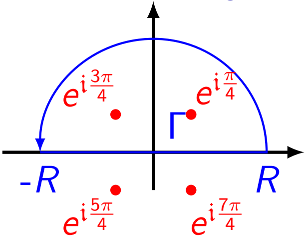


$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \text{Sum of residues at the}$$

poles of $\frac{1}{1+z^4}$ in the upper half plane.

Applications of the residue theorem

Calculation of real integrals



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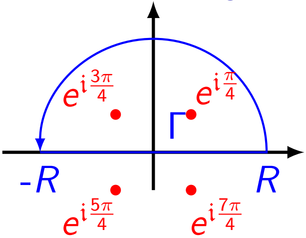
The poles of $\frac{1}{1+z^4}$ are at

$$z_k = e^{i\frac{(2k-1)\pi}{4}}, \quad k = 1, 2, 3, 4$$

Applications of the residue theorem

Calculation of real integrals



$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} =$ Sum of residues at the poles of $\frac{1}{1+z^4}$ in the upper half plane.

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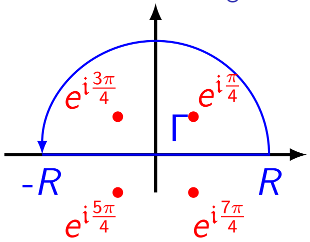

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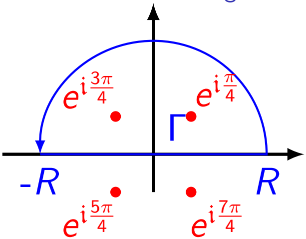
of which only $e^{i\pi/4}$ and $e^{i3\pi/4}$ are in the upper half plane.

$$\operatorname{Res}_{z=e^{i\pi/4}} \frac{1}{1+z^4} = \lim_{z\to e^{i\pi/4}} \left(z - e^{i\frac{\pi}{4}}\right) \frac{1}{1+z^4}$$

Applications of the residue theorem

Calculation of real integrals



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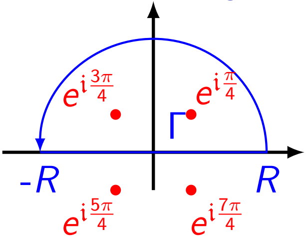
$$\text{Res}_{z=e^{i\pi/4}} \frac{1}{1+z^4} = \lim_{z \to e^{i\pi/4}} \frac{1}{4z^3}$$

Applications of the residue theorem

Calculation of real integrals



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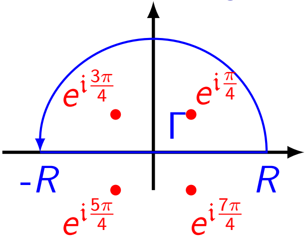
Applications of the residue theorem

Calculation of real integrals



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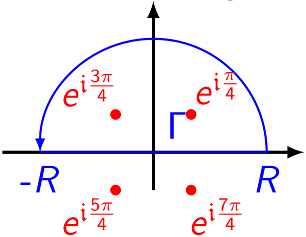
Applications of the residue theorem

Calculation of real integrals



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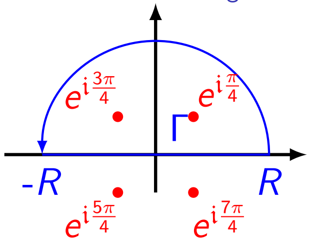
of which only $e^{i\pi/4}$ and $e^{i3\pi/4}$ are in the upper half plane.

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} =$$

Applications of the residue theorem

Calculation of real integrals



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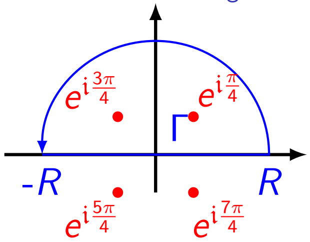
Applications of the residue theorem

Calculation of real integrals



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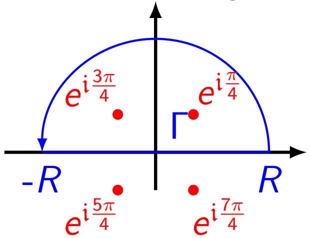
Applications of the residue theorem

Calculation of real integrals



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$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}$$

Applications of the residue theorem

Calculation of real integrals

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2 + b^2} dx$$

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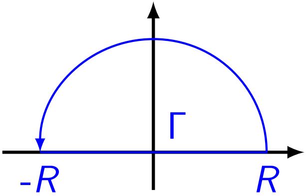

Consider the integral

$$\oint_\Gamma \frac{e^{iaz}}{z^2+b^2} dz, \;\; a>0, b>0$$

Applications of the residue theorem

Calculation of real integrals

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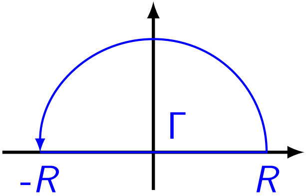

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Applications of the residue theorem

Calculation of real integrals

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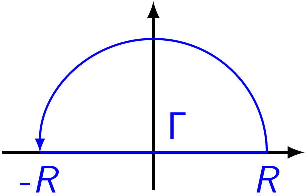

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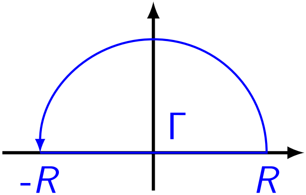

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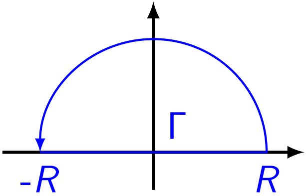
Of the two poles, only $+ib$ is in the upper half plane

Applications of the residue theorem

Calculation of real integrals

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2 + b^2} dx$$



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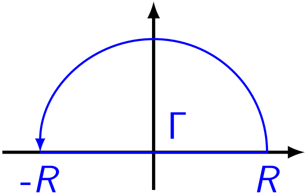
Applications of the residue theorem

Calculation of real integrals

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+b^2}\,dx$$



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Applications of the residue theorem

Calculation of real integrals

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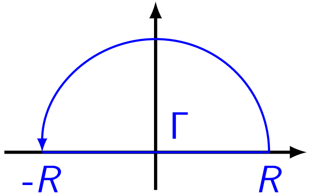

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Applications of the residue theorem

Calculation of real integrals

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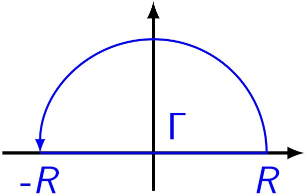

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Applications of the residue theorem

Calculation of real integrals

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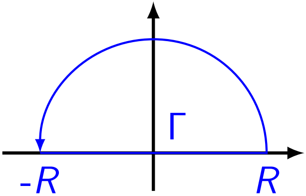

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Applications of the residue theorem

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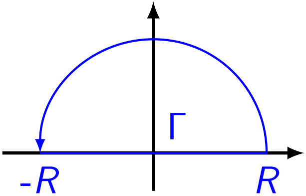

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Applications of the residue theorem

Calculation of real integrals

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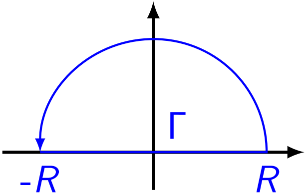

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- $f(z)$ is analytic on an open set containing the region $\Im(z) \geq 0$ (the real axis and the upper half plane), except possibly for a finite number of isolated singular points.

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- The only singularities of $f(z)$ on the real axis coincides with the zeros of $\sin x$ (or $\cos x$) and are poles of order k or less.

Applications of the residue theorem

Calculation of real integrals - case 2

$$\int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3} dx$$

Applications of the residue theorem

Calculation of real integrals - case 2

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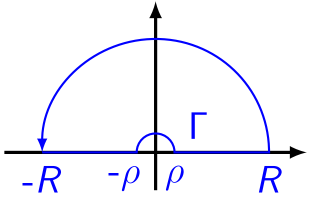
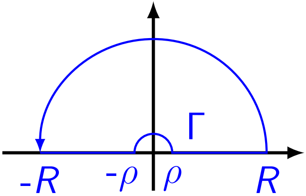


Since $\sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x$

Applications of the residue theorem

Calculation of real integrals - case 2

$$\int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3} dx$$



Since $\sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x$ consider the integral

$$\oint_\Gamma \frac{\frac{3}{4} e^{iz} - \frac{1}{4} e^{i3z}}{z^3} dz$$

Applications of the residue theorem

Calculation of real integrals - case 2

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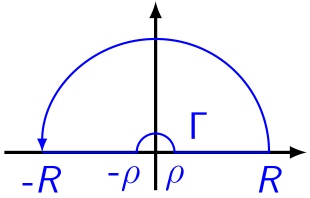

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Applications of the residue theorem

Calculation of real integrals - case 2

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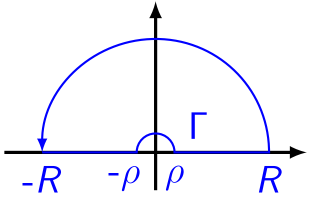

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Applications of the residue theorem

Calculation of real integrals - case 2

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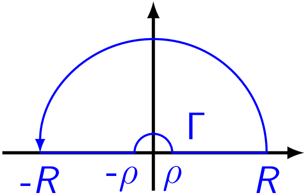

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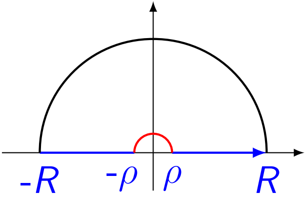
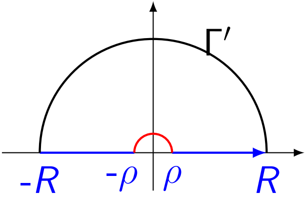
Applications of the residue theorem

Calculation of real integrals - case 2



The imaginary part of the sum of the two integrals over the two straight line pieces is our answer in the limit $R \longrightarrow \infty$ and $\rho \longrightarrow 0$

$$\int_{-R}^{-\rho} \frac{\frac{3}{4}e^{ix} - \frac{1}{4}e^{i3x}}{x^3} dx + \int_{\rho}^{R} \frac{\frac{3}{4}e^{ix} - \frac{1}{4}e^{i3x}}{x^3} dx$$

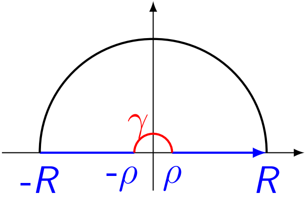


The integral over the larger semicircle vanishes.

$$\int_{\Gamma'} \frac{\frac{3}{4}e^{iz} - \frac{1}{4}e^{i3z}}{z^3} dz$$

Applications of the residue theorem

Calculation of real integrals - case 2



We need the value of the integral over the smaller semicircle, in the limit $\rho \to 0$

$$\int_\gamma \frac{\frac{3}{4}e^{iz} - \frac{1}{4}e^{i3z}}{z^3} dz$$

Applications of the residue theorem

Calculation of real integrals - case 2

$$\lim_{\rho \to 0} \int_{\gamma} \frac{\frac{3}{4}e^{iz} - \frac{1}{4}e^{i3z}}{z^3} dz$$

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Since we are integrating over a very small semicircle, we look at the Laurent expansion of the integrand.

$$\lim_{\rho\to 0}\int_\gamma \frac{\frac{3}{4}e^{iz}-\frac{1}{4}e^{i3z}}{z^3}dz$$

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$$\begin{aligned}\frac{\frac{3}{4}e^{iz}-\frac{1}{4}e^{i3z}}{z^3} &= \frac{1}{z^3}\left[\frac{3}{4}\left(1+iz+i^2\frac{z^2}{2!}+i^3\frac{z^3}{3!}+\dots\right)\right.\\ &- \left.\frac{1}{4}\left(1+i3z+i^2\frac{3^2z^2}{2!}+i^3\frac{3^3z^3}{3!}+\dots\right)\right]\end{aligned}$$

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Since we are integrating over a very small semicircle, we look at the Laurent expansion of the integrand.

$$\begin{aligned} \frac{\frac{3}{4} e^{iz} - \frac{1}{4} e^{i3z}}{z^3} &= \frac{1}{z^3} \left[\frac{3}{4} \left(1 + iz + i^2 \frac{z^2}{2!} + i^3 \frac{z^3}{3!} + \dots \right) \right. \\ &- \left. \frac{1}{4} \left(1 + i3z + i^2 \frac{3^2 z^2}{2!} + i^3 \frac{3^3 z^3}{3!} + \dots \right) \right] \\ &= \frac{1}{2z^3} + \frac{3}{4z} + g(z) \end{aligned}$$

where $g(z)$ is analytic at $z = 0$.

Applications of the residue theorem

Calculation of real integrals - case 2

$$\lim_{\rho\to 0}\int_\gamma \frac{\frac{3}{4}e^{iz}-\frac{1}{4}e^{i3z}}{z^3}dz = \lim_{\rho\to 0}\left[\frac{1}{2}\underbrace{\int_\gamma \frac{dz}{z^3}}_{I_1}+\frac{3}{4}\underbrace{\int_\gamma \frac{dz}{z}}_{I_2}+\underbrace{\int_\gamma g(z)dz}_{I_3}\right]$$

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$g(z)$ is analytic at $z=0$ implies that it is bounded in an open disc centered at 0. $z\in\gamma \implies |g(z)|<M$

Applications of the residue theorem

Calculation of real integrals - case 2

$$\lim_{\rho\to 0}\int_\gamma \frac{\frac{3}{4}e^{iz}-\frac{1}{4}e^{i3z}}{z^3}dz = \lim_{\rho\to 0}\left[\frac{1}{2}\underbrace{\int_\gamma \frac{dz}{z^3}}_{I_1}+\frac{3}{4}\underbrace{\int_\gamma \frac{dz}{z}}_{I_2}+\underbrace{\int_\gamma g(z)dz}_{I_3}\right]$$

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Applications of the residue theorem

Calculation of real integrals - case 2

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Applications of the residue theorem

Calculation of real integrals - case 2

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Applications of the residue theorem

Calculation of real integrals - case 2

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Applications of the residue theorem

Calculation of real integrals - case 2

$$\lim_{\rho\to 0}\int_\gamma \frac{\frac{3}{4}e^{\mathrm{i}z}-\frac{1}{4}e^{\mathrm{i}3z}}{z^3}dz = \lim_{\rho\to 0}\left[\frac{1}{2}\underbrace{\int_\gamma \frac{dz}{z^3}}_{I_1}+\frac{3}{4}\underbrace{\int_\gamma \frac{dz}{z}}_{I_2}+\underbrace{\int_\gamma g(z)dz}_{I_3}\right]$$

$$I_1=\int_\pi^0 \frac{\mathrm{i}\rho e^{\mathrm{i}\theta}d\theta}{\rho^3 e^{\mathrm{i}3\theta}}$$

Applications of the residue theorem

Calculation of real integrals - case 2

$$\lim_{\rho\to 0}\int_\gamma \frac{\frac{3}{4}e^{iz}-\frac{1}{4}e^{i3z}}{z^3}dz = \lim_{\rho\to 0}\left[\frac{1}{2}\underbrace{\int_\gamma \frac{dz}{z^3}}_{I_1}+\frac{3}{4}\underbrace{\int_\gamma \frac{dz}{z}}_{I_2}+\underbrace{\int_\gamma g(z)dz}_{I_3}\right]$$

$$I_1 = \frac{i}{\rho^2}\int_\pi^0 e^{-i2\theta}d\theta$$

Applications of the residue theorem

Calculation of real integrals - case 2

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$$I_1 = \frac{i}{\rho^2}\,\frac{e^{-2i\pi}-1}{-2i\theta}$$

Applications of the residue theorem

Calculation of real integrals - case 2

$$\lim_{\rho\to 0}\int_\gamma \frac{\frac{3}{4}e^{iz}-\frac{1}{4}e^{i3z}}{z^3}dz = \lim_{\rho\to 0}\left[\frac{1}{2}\underbrace{\int_\gamma \frac{dz}{z^3}}_{I_1}+\frac{3}{4}\underbrace{\int_\gamma \frac{dz}{z}}_{I_2}+\underbrace{\int_\gamma g(z)dz}_{I_3}\right]$$

$$I_1 = 0$$

Applications of the residue theorem

Calculation of real integrals - case 2

$$\lim_{\rho\to 0}\int_\gamma \frac{\frac{3}{4}e^{iz}-\frac{1}{4}e^{i3z}}{z^3}dz=\lim_{\rho\to 0}\left[\frac{1}{2}\underbrace{\int_\gamma \frac{dz}{z^3}}_{I_1}+\frac{3}{4}\underbrace{\int_\gamma \frac{dz}{z}}_{I_2}+\underbrace{\int_\gamma g(z)dz}_{I_3}\right]$$

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Applications of the residue theorem

Calculation of real integrals - case 2

$$\lim_{\rho\to 0}\int_\gamma \frac{\frac{3}{4}e^{iz}-\frac{1}{4}e^{i3z}}{z^3}dz = \lim_{\rho\to 0}\left[\frac{1}{2}\underbrace{\int_\gamma \frac{dz}{z^3}}_{I_1}+\frac{3}{4}\underbrace{\int_\gamma \frac{dz}{z}}_{I_2}+\underbrace{\int_\gamma g(z)dz}_{I_3}\right]$$

$$I_2 = \int_\pi^0 \frac{i\rho e^{i\theta}d\theta}{\rho e^{i\theta}}$$

Applications of the residue theorem

Calculation of real integrals - case 2

$$\lim_{\rho\to 0}\int_\gamma \frac{\frac{3}{4}e^{iz}-\frac{1}{4}e^{i3z}}{z^3}dz = \lim_{\rho\to 0}\left[\frac{1}{2}\underbrace{\int_\gamma \frac{dz}{z^3}}_{I_1}+\frac{3}{4}\underbrace{\int_\gamma \frac{dz}{z}}_{I_2}+\underbrace{\int_\gamma g(z)dz}_{I_3}\right]$$

$$I_2 = i\int_\pi^0 d\theta$$

Applications of the residue theorem

Calculation of real integrals - case 2

$$\lim_{\rho\to 0}\int_\gamma \frac{\frac{3}{4}e^{iz}-\frac{1}{4}e^{i3z}}{z^3}dz = \lim_{\rho\to 0}\left[\frac{1}{2}\underbrace{\int_\gamma \frac{dz}{z^3}}_{I_1}+\frac{3}{4}\underbrace{\int_\gamma \frac{dz}{z}}_{I_2}+\underbrace{\int_\gamma g(z)dz}_{I_3}\right]$$

$$I_2 = -i\pi$$

Applications of the residue theorem

Calculation of real integrals - case 2

$$\lim_{\rho\to 0}\int_\gamma \frac{\frac{3}{4}e^{iz}-\frac{1}{4}e^{i3z}}{z^3}dz = \lim_{\rho\to 0}\left[\frac{1}{2}\underbrace{\int_\gamma \frac{dz}{z^3}}_{I_1}+\frac{3}{4}\underbrace{\int_\gamma \frac{dz}{z}}_{I_2}+\underbrace{\int_\gamma g(z)dz}_{I_3}\right]$$

$$\lim_{\rho\to 0}\int_\gamma \frac{\frac{3}{4}e^{iz}-\frac{1}{4}e^{i3z}}{z^3}dz = -i\frac{3\pi}{4}$$

Applications of the residue theorem

Calculation of real integrals - case 2

$$\lim_{\substack{R\to\infty \\ \rho\to 0}} \left[\int_{-R}^{-\rho} \frac{\frac{3}{4}e^{\mathrm{i}x} - \frac{1}{4}e^{\mathrm{i}3x}}{x^3}dx + \int_{\rho}^{R} \frac{\frac{3}{4}e^{\mathrm{i}x} - \frac{1}{4}e^{\mathrm{i}3x}}{x^3}dx\right] = \mathrm{i}\frac{3\pi}{4}$$

Applications of the residue theorem

Calculation of real integrals - case 2

$$\lim_{\substack{R\to\infty \\ \rho\to 0}} \left[\int_{-R}^{-\rho} \frac{\frac{3}{4}e^{ix} - \frac{1}{4}e^{i3x}}{x^3} dx + \int_{\rho}^{R} \frac{\frac{3}{4}e^{ix} - \frac{1}{4}e^{i3x}}{x^3} dx \right] = i\frac{3\pi}{4}$$

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\frac{3}{4}e^{ix} - \frac{1}{4}e^{i3x}}{x^3} dx = i\frac{3\pi}{4}$$

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Taking imaginary parts of both sides give us

$$\int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3} dx = \frac{3\pi}{4}$$

Consider integrals of the form

$$\int_{-\infty}^{\infty} f(x)\sin x dx = 2\int_{0}^{\infty} f(x)\sin x dx$$

where

- $f(x)$ is an odd function of x.

Applications of the residue theorem

Calculation of real integrals - case 3

Consider integrals of the form

$$\int_{-\infty}^{\infty} f(x)\sin x dx = 2\int_{0}^{\infty} f(x)\sin x dx$$

where

- $f(x)$ is an odd function of x.
- Except for at most a finite number of singular points, $f(z)$ is analytic in an open set containing the real axis and the upper half plane.

Applications of the residue theorem

Calculation of real integrals - case 3

Consider integrals of the form

$$\int_{-\infty}^{\infty} f(x) \sin x dx = 2 \int_{0}^{\infty} f(x) \sin x dx$$

where

- $f(x)$ is an odd function of x.
- Except for at most a finite number of singular points, $f(z)$ is analytic in an open set containing the real axis and the upper half plane.
- All singularities of $f(z)$ on the real axis are simple poles coinciding with the zeroes of $\sin x$.

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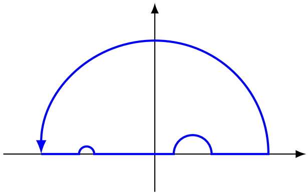
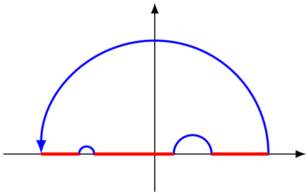
- Except for at most a finite number of singular points, $f(z)$ is analytic in an open set containing the real axis and the upper half plane.
- All singularities of $f(z)$ on the real axis are simple poles coinciding with the zeroes of $\sin x$.
- $|zf(z)| \to 0$ as $|z| \to \infty$ in the upper half plane.

For this we integrate

$$\oint_{\Gamma} f(z)e^{iz}dz$$

where the contour Γ is composed of



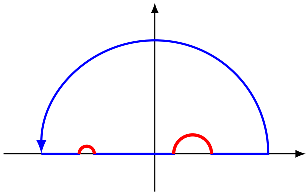


For this we integrate

$$\oint_\Gamma f(z)e^{\mathrm{i}z}dz$$

where the contour Γ is composed of

- Straight line segments along the real axis.



For this we integrate

$$\oint_{\Gamma} f(z) e^{iz} dz$$

where the contour Γ is composed of

- Straight line segments along the real axis.
- Small semicircular indentations to avoid singular points of $f(z)$ on the real axis, if any.

For this we integrate

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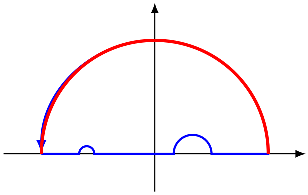

- Straight line segments along the real axis.
- Small semicircular indentations to avoid singular points of $f(z)$ on the real axis, if any.
- Large semicircle in the upper half plane.

Jordan's lemma

Under the conditions stated on $f(z)$, the integral $\int_{\Gamma'} f(z)e^{iz}dz$, where Γ' is the semicircle of radius R in the upper half plane centered at the origin, vanishes in the limit $R \to \infty$.

Jordan's lemma

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Using the parameterization $z(\theta) = Re^{i\theta}, 0 \leq \theta \leq \pi$,

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Using the parameterization $z(\theta) = Re^{i\theta}, 0 \leq \theta \leq \pi$,

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Under the conditions stated on $f(z)$, the integral $\int_{\Gamma'} f(z)e^{iz}dz$, where Γ' is the semicircle of radius R in the upper half plane centered at the origin, vanishes in the limit $R \to \infty$.

Using the parameterization $z(\theta) = Re^{i\theta}, 0 \leq \theta \leq \pi$,

$$\left| \int_{\Gamma'} f(z)e^{iz}dz \right| \leq R \int_0^{\pi} \left| f\left(Re^{i\theta}\right) \right| e^{-R\sin\theta} d\theta$$

Jordan's lemma

Under the conditions stated on $f(z)$, the integral $\int_{\Gamma'} f(z)e^{iz}dz$, where Γ' is the semicircle of radius R in the upper half plane centered at the origin, vanishes in the limit $R \to \infty$.

Using the parameterization $z(\theta) = Re^{i\theta}, 0 \le \theta \le \pi$,

$$\left| \int_{\Gamma'} f(z)e^{iz}dz \right| \le R \int_0^{\pi} \frac{M}{R} e^{-R\sin\theta} d\theta$$

$\because \lim_{|z|\to\infty} |zf(z)| = 0$, for any $M > 0$, $\exists \rho > 0$ such that $|zf(z)| < M$ for $|z| > \rho$. Choose $R > \rho$.

Jordan's lemma

Under the conditions stated on $f(z)$, the integral $\int_{\Gamma'} f(z)e^{iz}dz$, where Γ' is the semicircle of radius R in the upper half plane centered at the origin, vanishes in the limit $R \to \infty$.

Using the parameterization $z(\theta) = Re^{i\theta}, 0 \leq \theta \leq \pi$,

$$\left| \int_{\Gamma'} f(z)e^{iz}dz \right| \leq M \int_0^{\pi} e^{-R\sin\theta} d\theta$$

Jordan's lemma

Under the conditions stated on $f(z)$, the integral $\int_{\Gamma'} f(z)e^{iz}dz$, where Γ' is the semicircle of radius R in the upper half plane centered at the origin, vanishes in the limit $R \to \infty$.

Using the parameterization $z(\theta) = Re^{i\theta}, 0 \leq \theta \leq \pi$,

$$\left| \int_{\Gamma'} f(z)e^{iz}dz \right| \leq \frac{\pi}{R}\left(1 - e^{-R}\right)$$

(From Jordan's inequality.)

Why?

Jordan's lemma

Under the conditions stated on $f(z)$, the integral $\int_{\Gamma'} f(z)e^{iz}dz$, where Γ' is the semicircle of radius R in the upper half plane centered at the origin, vanishes in the limit $R \to \infty$.

Using the parameterization $z(\theta) = Re^{i\theta}, 0 \leq \theta \leq \pi$,

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Using the parameterization $z(\theta) = Re^{i\theta}, 0 \leq \theta \leq \pi$,

$$\lim_{R\to\infty} \left| \int_{\Gamma'} f(z)e^{iz}dz \right| \leq 0$$

Applications of the residue theorem

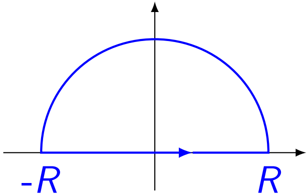
Calculation of real integrals - case 3

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1} dx$$

Applications of the residue theorem

Calculation of real integrals - case 3

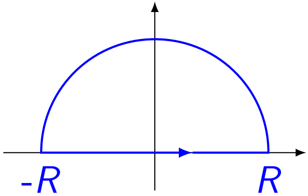
$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1} dx$$



Consider the integral

$$\oint_{\Gamma} \frac{ze^{iz}}{z^2+1} dz$$

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1} dx$$



Consider the integral

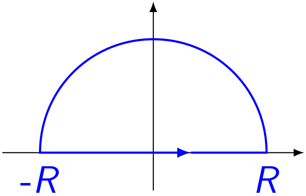
$$\oint_{\Gamma} \frac{ze^{iz}}{z^2+1} dz$$

The only singularity in the upper half plane is $z = \imath$.

Applications of the residue theorem

Calculation of real integrals - case 3

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1} dx$$



Consider the integral

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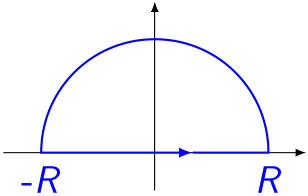
$$\mathrm{Res}_{z=+\mathrm{i}} \frac{ze^{iz}}{z^2+1} = \lim_{z \to \mathrm{i}} \frac{ze^{iz}(z-\mathrm{i})}{z^2+1}$$

Applications of the residue theorem

Calculation of real integrals - case 3

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1} dx$$



Consider the integral

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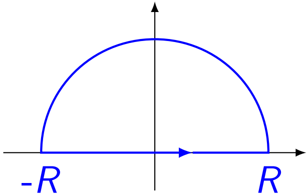
$$\mathrm{Res}_{z=+\imath} \frac{ze^{iz}}{z^2+1} = \lim_{z\to\imath} \frac{ze^{iz}(z-\imath)}{z^2+1} = \lim_{z\to\imath} \frac{ze^{iz}}{z+\imath}$$

Applications of the residue theorem

Calculation of real integrals - case 3

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1} dx$$



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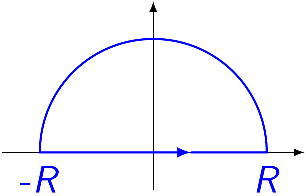
Applications of the residue theorem

Calculation of real integrals - case 3

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1} dx$$



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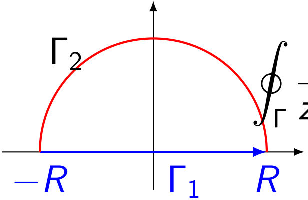
Thus by residue theorem

$$\lim_{R\to\infty} \oint_{\Gamma} \frac{ze^{iz}}{z^2+1} dz = \frac{\mathrm{i}\pi}{e}$$

Applications of the residue theorem

Calculation of real integrals - case 3

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1} dx$$

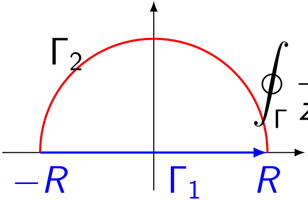


$$\oint_{\Gamma} \frac{ze^{iz}}{z^2+1} dz = \int_{\Gamma_1} \frac{ze^{iz}}{z^2+1} dz + \int_{\Gamma_2} \frac{ze^{iz}}{z^2+1} dz$$

Applications of the residue theorem

Calculation of real integrals - case 3

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1} dx$$



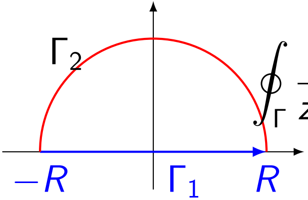
$$\oint_{\Gamma} \frac{ze^{iz}}{z^2+1} dz = \int_{\Gamma_1} \frac{ze^{iz}}{z^2+1} dz + \int_{\Gamma_2} \frac{ze^{iz}}{z^2+1} dz$$

By Jordan's lemma the second integral $\to 0$ as $R \to \infty$.

Applications of the residue theorem

Calculation of real integrals - case 3

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1} dx$$



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$$\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+1} dz = \frac{i\pi}{e}$$

Applications of the residue theorem

Calculation of real integrals - case 3

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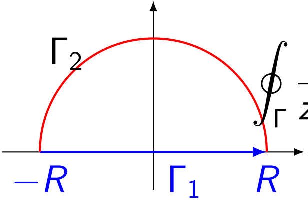

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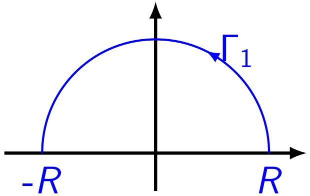
$$\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+1} dz = \frac{i\pi}{e}$$

Taking the imaginary part of both sides :

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1} dz = \frac{\pi}{e}$$

Proof that $\lim_{R\to\infty} \int_{\Gamma_1} f(z)dz = 0$



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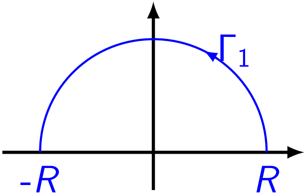

Using the parameterization $z(\theta) = Re^{i\theta},\ 0 \leq \theta \leq \pi$ we get

$$\int_{\Gamma_1} f(z)dz = iR \int_0^{\pi} f\left(Re^{i\theta}\right) d\theta$$

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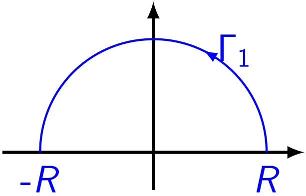

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Since $f(z)|z|^{\alpha} \to 0$ as $|z| \to \infty$, for a given $M > 0$,

$$\exists \rho > 0 \ :$$

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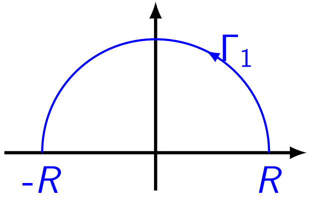

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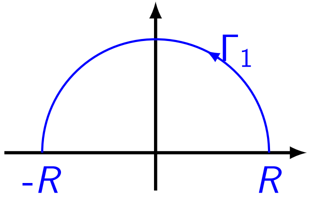
Proof that $\lim_{R\to\infty} \int_{\Gamma_1} f(z)dz = 0$



Using the parameterization $z(\theta) = Re^{i\theta}$, $0 \le \theta \le \pi$ we get

$$\int_{\Gamma_1} f(z)dz = iR \int_0^{\pi} f\left(Re^{i\theta}\right) d\theta$$

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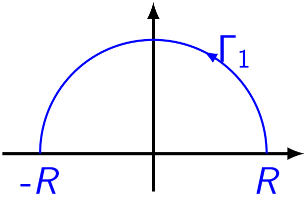
Proof that $\lim_{R\to\infty} \int_{\Gamma_1} f(z)dz = 0$



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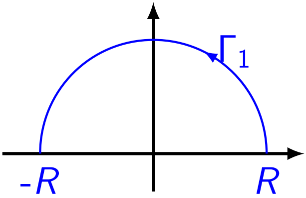

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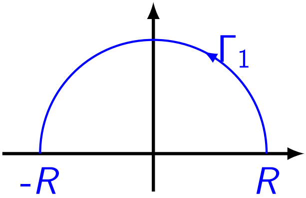

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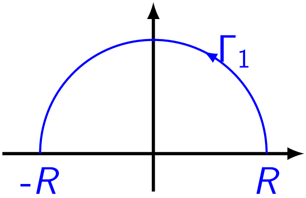

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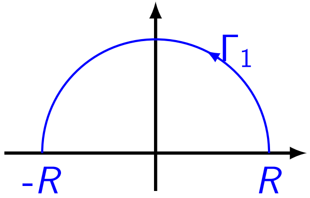

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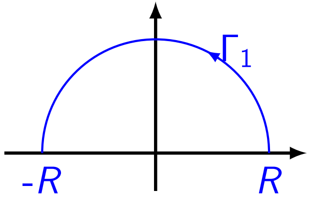

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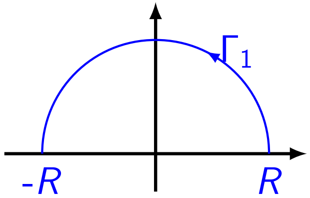

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Jordan's inequality

$$\int_0^\pi e^{-R\sin\theta}d\theta \leq \frac{\pi}{R}\left(1 - e^{-R}\right), \; R > 0$$

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- we have

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$$\begin{aligned}
\int_0^{\pi} e^{-R\sin\theta} d\theta &= 2\int_0^{\frac{\pi}{2}} e^{-R\sin\theta} d\theta \leq 2\int_0^{\frac{\pi}{2}} e^{-\frac{2R\theta}{\pi}} d\theta \\
&= \frac{\pi}{R}\left(1 - e^{-R}\right)
\end{aligned}$$