Further applications of the Residue theorem

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MA211, Lecture 25

Integrands with branch points

$$\int_0^\infty x^\alpha f(x)dx, \qquad 0 > \alpha > -1$$

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- $f(z) = \mathcal{O}\left(z^{-1}\right)$, as $|z| \to \infty$.
- This is most commonly used for integrals of the type

$$\int_0^\infty x^\alpha \frac{P(x)}{Q(x)} dx, \qquad \deg\left(P(x)\right) < \deg\left(Q(x)\right)$$

  where $P(x)$ and $Q(x)$ are polynomials

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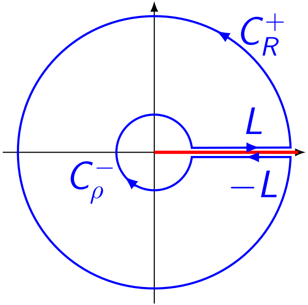

Consider the integral

$$\int_C \frac{z^\alpha}{z^2+1}dz$$

where $C = C_R^+ \cup (-L) \cup C_\rho^- \cup L$.

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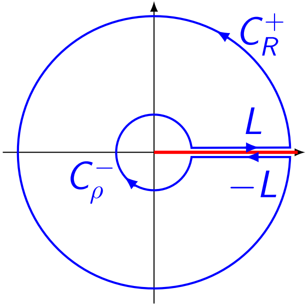

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To make the integrand single-valued, we choose the branch

$$z = re^{i\theta} \mapsto z^\alpha = r^\alpha e^{i\alpha\theta}, \; 0 < \theta \le 2\pi$$

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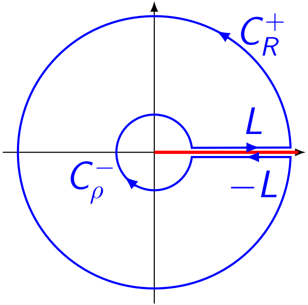

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To make the integrand single-valued, we choose the branch

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so that the branch cut joins the branch points 0 and ∞ along the positive real axis.

Integrands with branch points

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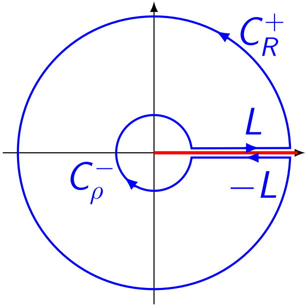

$$\int_{C_R^+} \frac{z^\alpha}{z^2+1}dz = \mathrm{i}\int_0^{2\pi} \frac{R^{\alpha+1}e^{\mathrm{i}(\alpha+1)\theta}}{R^2e^{\mathrm{i}2\theta}+1}d\theta$$

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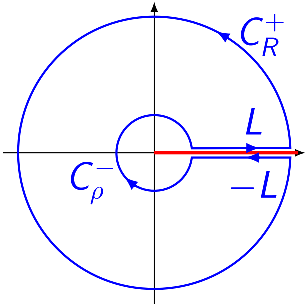

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$$\left|\int_{C_R^+} \frac{z^\alpha}{z^2+1}dz\right| \le \int_0^{2\pi} \frac{R^{\alpha+1}d\theta}{R^2-1}$$

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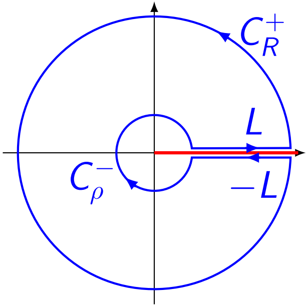

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$$\left|\int_{C_R^+} \frac{z^\alpha}{z^2+1}dz\right| \le \frac{2\pi R^{\alpha+1}}{R^2-1}$$

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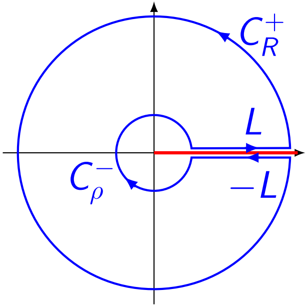

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$$\left|\int_{C_R^+} \frac{z^\alpha}{z^2+1}dz\right| \le \frac{2\pi R^{\alpha+1}}{R^2-1} \underset{R\to\infty}{\longrightarrow} 0$$

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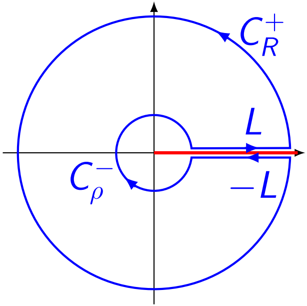

Similarly

$$\left|\int_{C_\rho^-} \frac{z^\alpha}{z^2+1}dz\right| \leq \frac{2\pi\rho^{\alpha+1}}{\rho^2-1} \underset{\rho\to 0}{\longrightarrow} 0$$

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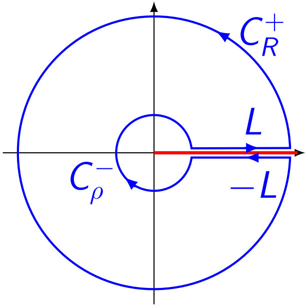

$$\lim_{\substack{R\to\infty \\ \rho\to 0}} \int_L f(z)dz = \int_0^\infty f(x+i\epsilon)dx$$

$$\lim_{\substack{R\to\infty \\ \rho\to 0}} \int_{-L} f(z)dz = \int_\infty^0 f(x-i\epsilon)dx$$

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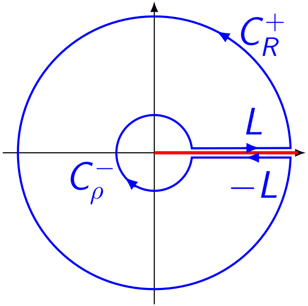

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Because of the discontinuity of the branch $f(z)$ across the branch cut, the two integrals above do not cancel!

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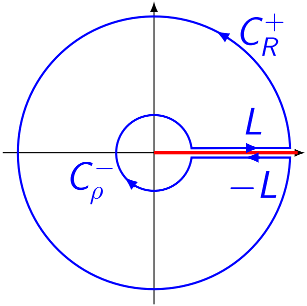

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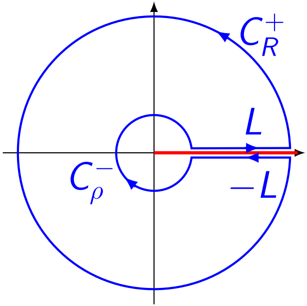

$$\int_C \frac{z^{\alpha}}{z^2+1} dz = \left(1 - e^{i2\pi\alpha}\right) \int_0^{\infty} \frac{x^{\alpha}}{x^2+1} dx$$

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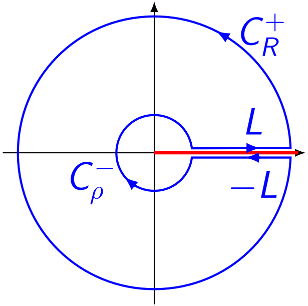

$$\int_C \frac{z^\alpha}{z^2+1}dz = \left(1 - e^{i2\pi\alpha}\right)\int_0^\infty \frac{x^\alpha}{x^2+1}dx$$

calculating the contour integral on the left will yield the value of the real integral that we want.

Integrands with branch points

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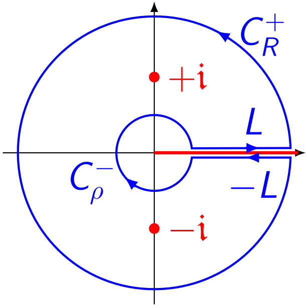

The two poles of the integrand are at $z = \pm i$.

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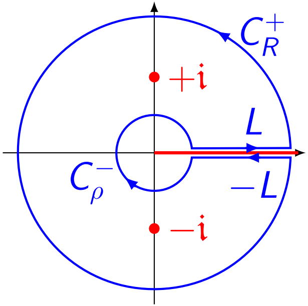

The two poles of the integrand are at $z = \pm i$.
To calculate the residues at these poles, we have to keep the choice of the branch of the function in mind.

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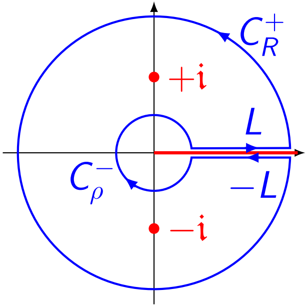

The two poles of the integrand are at $z = \pm i$.
To calculate the residues at these poles, we have to keep the choice of the branch of the function in mind. Thus

$$+i = e^{i\frac{\pi}{2}}, \qquad -i = e^{i\frac{3\pi}{2}}$$

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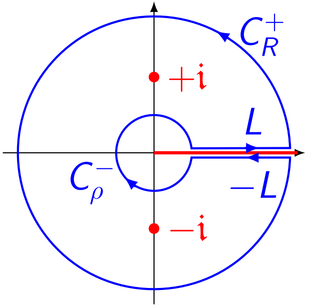

$$\operatorname*{Res}_{z=e^{i\frac{\pi}{2}}} \frac{z^\alpha}{z^2+1} = \lim_{z\to e^{i\frac{\pi}{2}}} \frac{z^\alpha\left(z-e^{i\frac{\pi}{2}}\right)}{z^2+1}$$

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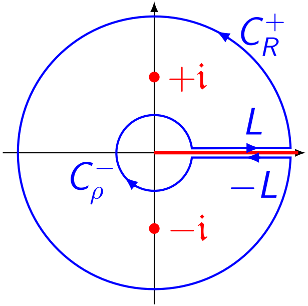

$$\operatorname*{Res}_{z=e^{\mathrm{i}\frac{\pi}{2}}} \frac{z^\alpha}{z^2+1} = e^{\mathrm{i}\frac{\pi\alpha}{2}} \lim_{z\to e^{\mathrm{i}\frac{\pi}{2}}} \frac{1}{2z}$$

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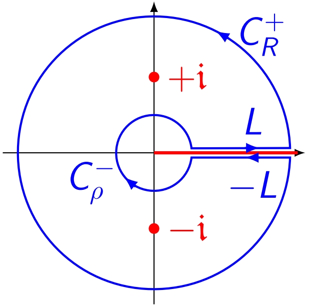

$$\operatorname*{Res}_{z=e^{i\frac{\pi}{2}}} \frac{z^\alpha}{z^2+1} = \frac{e^{i\frac{\pi\alpha}{2}}}{2i}$$

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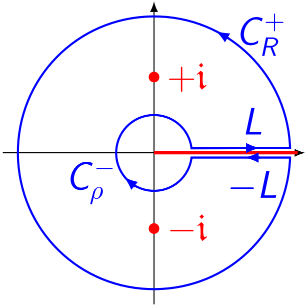

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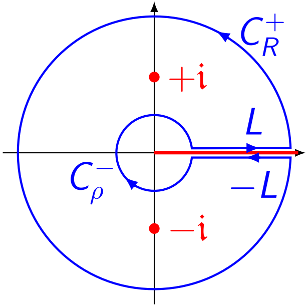

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$$\int_C \frac{z^\alpha}{z^2+1}dz = 2\pi i\left(\frac{e^{i\frac{\pi\alpha}{2}}}{2i} - \frac{e^{i\frac{3\pi\alpha}{2}}}{2i}\right)$$

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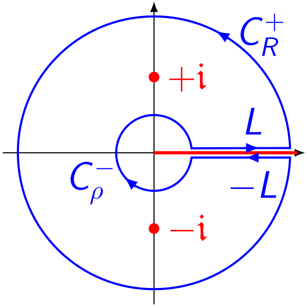

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$$\int_C \frac{z^\alpha}{z^2+1}dz = \pi e^{\mathrm{i}\frac{\pi\alpha}{2}}\left(1-e^{\mathrm{i}\pi\alpha}\right)$$

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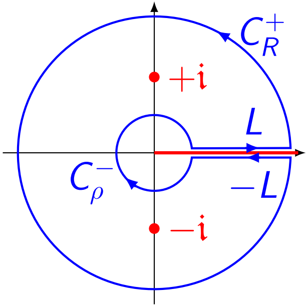

$$\left(1 - e^{\mathrm{i}2\pi\alpha}\right)\int_0^\infty \frac{x^\alpha}{x^2+1}dx$$
$$= \pi e^{\mathrm{i}\frac{\pi\alpha}{2}}\left(1 - e^{\mathrm{i}\pi\alpha}\right)$$

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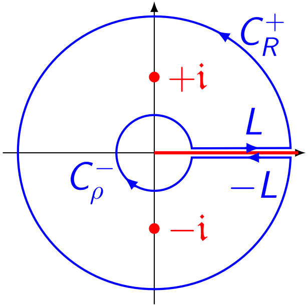

$$\int_0^\infty \frac{x^\alpha}{x^2+1}dx = \pi e^{\mathrm{i}\frac{\pi\alpha}{2}} \frac{1-e^{\mathrm{i}\pi\alpha}}{1-e^{\mathrm{i}2\pi\alpha}}$$

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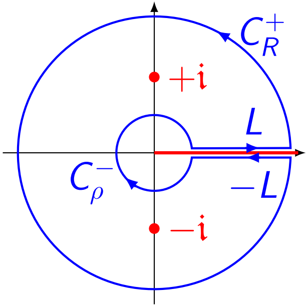

$$\int_0^\infty \frac{x^\alpha}{x^2+1}dx = \pi \frac{e^{\mathrm{i}\frac{\pi\alpha}{2}}}{1+e^{\mathrm{i}\pi\alpha}}$$

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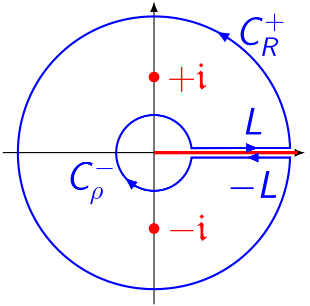

$$\int_0^\infty \frac{x^\alpha}{x^2+1}dx = \frac{\pi}{e^{i\frac{\pi\alpha}{2}} + e^{-i\frac{\pi\alpha}{2}}}$$

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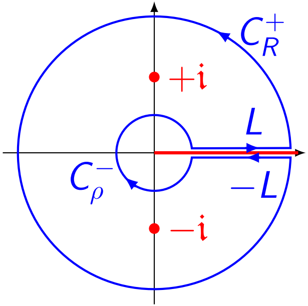

$$\int_0^\infty \frac{x^\alpha}{x^2+1}dx = \frac{\pi}{2\cos\left(\frac{\pi\alpha}{2}\right)}$$

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  - **Yes!**
  - You just have to be careful in chosing the argument for the complex numbers involved!
- Can I make use of any other branch cut?
  - **Yes!**
  - You must take a straight cut, though!

Integrands with logarithmic branch points

This method works for real integrals of the kind

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- $f(z) \to \mathcal{O}\left(z^{-2}\right)$ as $z \to \infty$

If $f(x)$ is an even function, we can convert the integral $\int_0^\infty f(x)dx$ to $\frac{1}{2}\int_{-\infty}^\infty f(x)dx$ and use the earlier method.

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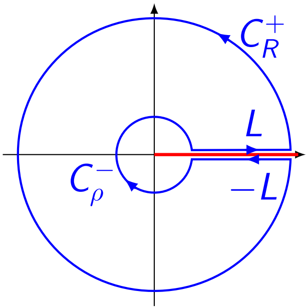
We will develop a **recursion formula** for this integral.

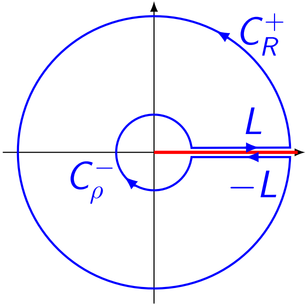


Integrands with logarithmic branch points

$$I_n = \int_0^\infty \frac{(\log x)^n}{(1+x)2} dx$$

We will develop a **recursion formula** for this integral.

Consider the integral



$$\oint_C \frac{(\log z)^N}{(1+z)^2} dz$$

Integrands with logarithmic branch points

$$I_n = \int_0^\infty \frac{(\log x)^n}{(1+x)2} dx$$

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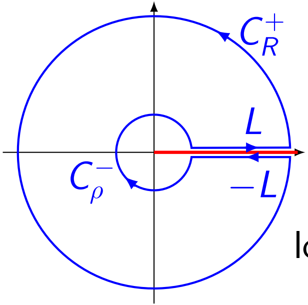

$$\oint_C \frac{(\log z)^N}{(1+z)^2} dz$$

and choose the branch

$$\log\ :\ z = re^{i\theta} \mapsto \log r + i\theta,\ \ 0 \le \theta < 2\pi$$

Integrands with logarithmic branch points

$$I_n = \int_0^\infty \frac{(\log x)^n}{(1+x)2} dx$$

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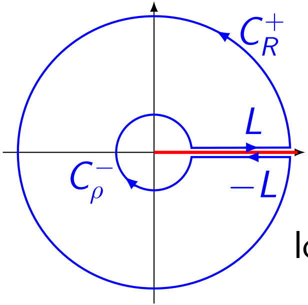


$$\oint_C \frac{(\log z)^N}{(1+z)^2} dz$$

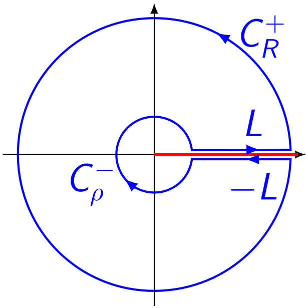
and choose the branch

$$\log : z = re^{i\theta} \mapsto \log r + i\theta, \ \ 0 \le \theta < 2\pi$$

Thus the branch cut joins the branch points 0 and ∞ along the positive real axis.

Integrands with logarithmic branch points

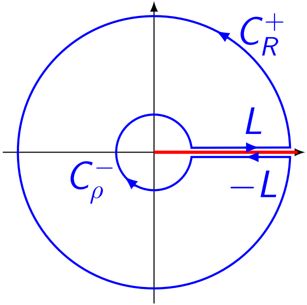
$$I_n = \int_0^\infty \frac{(\log x)^n}{(1+x)2} dx$$



The integrals over C_R^+ and C_ρ^- vanish in the limits $R \longrightarrow \infty$ and $\rho \longrightarrow 0$, respectively.

Integrands with logarithmic branch points

$$I_n = \int_0^\infty \frac{(\log x)^n}{(1+x)2} dx$$

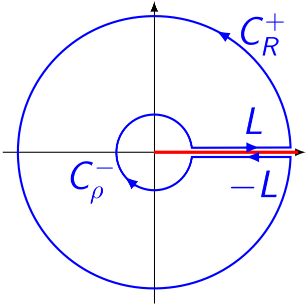


The integrals over C_R^+ and C_ρ^- vanish in the limits $R \longrightarrow \infty$ and $\rho \longrightarrow 0$, respectively.

The integrals over L and $-L$ do not cancel because of the discontinuity in $\log z$ across the branch cut.

Integrands with logarithmic branch points

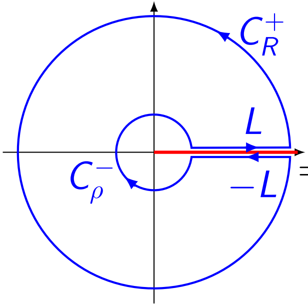
$$I_n = \int_0^\infty \frac{(\log x)^n}{(1+x)2} dx$$



$$\lim_{\substack{R\to\infty \\ \rho\to 0}} \oint_C \frac{(\log z)^N}{(1+z)^2} dz$$

Integrands with logarithmic branch points

$$I_n = \int_0^\infty \frac{(\log x)^n}{(1+x)2} dx$$

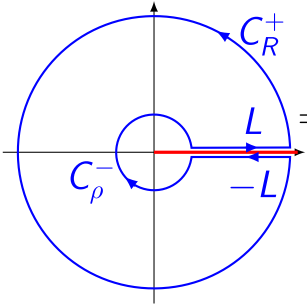


$$\lim_{\substack{R\to\infty \\ \rho\to 0}} \oint_C \frac{(\log z)^N}{(1+z)^2} dz$$

$$= \int_0^\infty \frac{(\log x)^N}{(1+x)^2} dx + \int_\infty^0 \frac{(\log x + 2\pi \mathrm{i})^N}{(1+x)^2} dx$$

Integrands with logarithmic branch points

$$I_n = \int_0^{\infty} \frac{(\log x)^n}{(1+x)2} dx$$



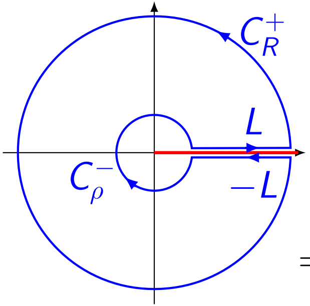
$$\lim_{\substack{R\to\infty \\ \rho\to 0}} \oint_C \frac{(\log z)^N}{(1+z)^2} dz$$

$$= \int_0^{\infty} \frac{(\log x)^N}{(1+x)^2} dx + \int_{\infty}^{0} \frac{(\log x + 2\pi \mathfrak{i})^N}{(1+x)^2} dx$$

$$= \int_0^{\infty} \frac{(\log x)^N - (\log x + 2\pi \mathfrak{i})^N}{(1+x)^2} dx$$

Integrands with logarithmic branch points

$$I_n = \int_0^\infty \frac{(\log x)^n}{(1+x)2} dx$$



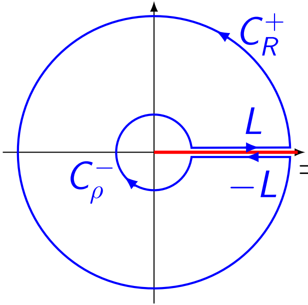
$$\lim_{\substack{R\to\infty \\ \rho\to 0}} \oint_C \frac{(\log z)^N}{(1+z)^2} dz$$

$$= \int_0^\infty \frac{(\log x)^N - (\log x + 2\pi \mathrm{i})^N}{(1+x)^2} dx$$

$$= -\int_0^\infty \frac{\sum_{k=0}^{N-1} \binom{N}{k} (2\pi \mathrm{i})^{N-k} (\log x)^k}{(1+x)^2} dx$$

Integrands with logarithmic branch points

$$I_n = \int_0^\infty \frac{(\log x)^n}{(1+x)2} dx$$



$$\lim_{\substack{R\to\infty \\ \rho\to 0}} \oint_C \frac{(\log z)^N}{(1+z)^2} dz$$

$$= -\sum_{k=0}^{N-1} \binom{N}{k} (2\pi \mathrm{i})^{N-k} \int_0^\infty \frac{(\log x)^k}{(1+x)^2} dx$$

Integrands with logarithmic branch points

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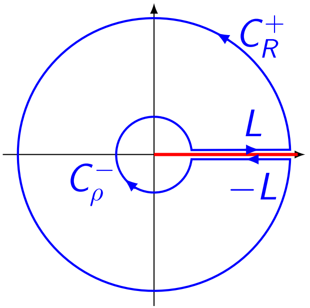
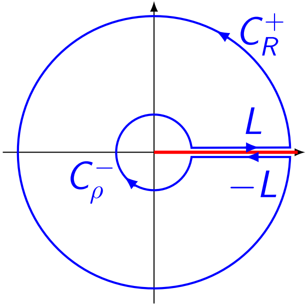


$$\lim_{\substack{R\to\infty \\ \rho\to 0}} \oint_C \frac{(\log z)^N}{(1+z)^2} dz$$

$$= -\sum_{k=0}^{N-1} \binom{N}{k} (2\pi \mathrm{i})^{N-k} I_k$$

Integrands with logarithmic branch points

$$I_n = \int_0^\infty \frac{(\log x)^n}{(1+x)2} dx$$



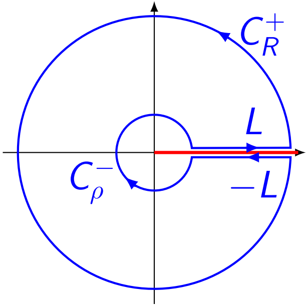
On the other hand it is easy to calculate the value of

$$\lim_{\substack{R\to\infty \\ \rho\to 0}} \oint_C \frac{(\log z)^N}{(1+z)^2} dz$$

using the Residue theorem.

Integrands with logarithmic branch points

$$I_n = \int_0^\infty \frac{(\log x)^n}{(1+x)2} dx$$



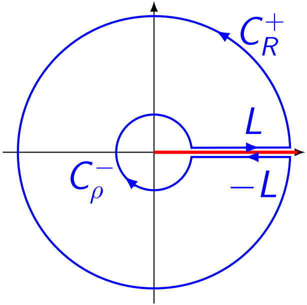
On the other hand it is easy to calculate the value of

$$\lim_{\substack{R\to\infty \\ \rho\to 0}} \oint_C \frac{(\log z)^N}{(1+z)^2} dz$$

using the Residue theorem.
One has to be careful, though, to use the correct branch ogf the log *function.*

Integrands with logarithmic branch points

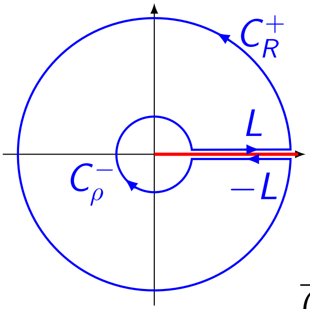
$$I_n = \int_0^{\infty} \frac{(\log x)^n}{(1+x)2} dx$$



The only singularity of the integrand inside the contour is a second order pole at $z = -1$.

Integrands with logarithmic branch points

$$I_n = \int_0^\infty \frac{(\log x)^n}{(1+x)2} dx$$



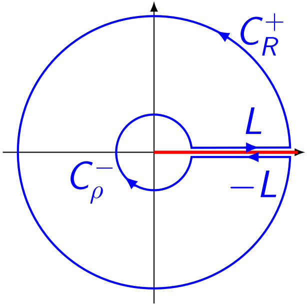
The only singularity of the integrand inside the contour is a second order pole at $z = -1$.

$$\operatorname*{Res}_{z=-1} \frac{(\log z)^N}{(1+z)^2} =$$

$$\frac{1}{(2-1)!} \frac{d^{2-1}}{dz^{2-1}} \left[(1+z)^2 \, \frac{(\log z)^N}{(1+z)^2} \right] \Bigg|_{z=-1}$$

Integrands with logarithmic branch points

$$I_n = \int_0^\infty \frac{(\log x)^n}{(1+x)2} dx$$



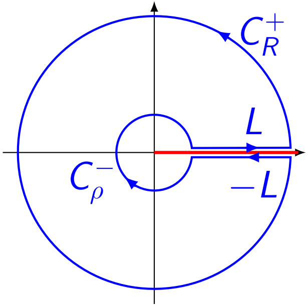
The only singularity of the integrand inside the contour is a second order pole at $z = -1$.

$$\operatorname*{Res}_{z=-1} \frac{(\log z)^N}{(1+z)^2} =$$

$$\frac{d}{dz}(\log z)^N \bigg|_{z=-1}$$

Integrands with logarithmic branch points

$$I_n = \int_0^\infty \frac{(\log x)^n}{(1+x)2} dx$$



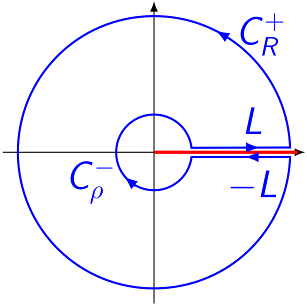
The only singularity of the integrand inside the contour is a second order pole at $z = -1$.

$$\operatorname*{Res}_{z=-1} \frac{(\log z)^N}{(1+z)^2} =$$

$$\frac{N}{z}(\log z)^{N-1}\Big|_{z=-1}$$

Integrands with logarithmic branch points

$$I_n = \int_0^\infty \frac{(\log x)^n}{(1+x)2} dx$$



The only singularity of the integrand inside the contour is a second order pole at $z = -1$.

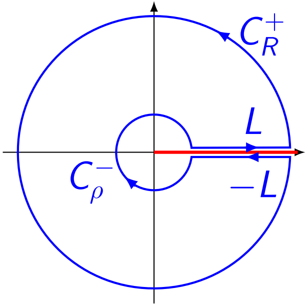
$$\operatorname*{Res}_{z=-1} \frac{(\log z)^N}{(1+z)^2} =$$

$$\left. \frac{N}{z} (\log z)^{N-1} \right|_{z=-1}$$

For the branch we have chosen, $\log(-1) = i\pi$

Integrands with logarithmic branch points

$$I_n = \int_0^\infty \frac{(\log x)^n}{(1+x)2} dx$$



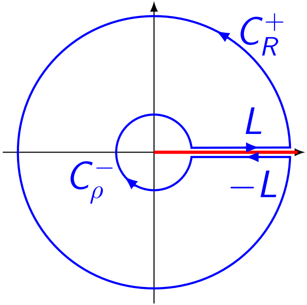
The only singularity of the integrand inside the contour is a second order pole at $z = -1$.

$$\operatorname*{Res}_{z=-1} \frac{(\log z)^N}{(1+z)^2} =$$

$$-N(i\pi)^{N-1}$$

Integrands with logarithmic branch points

$$I_n = \int_0^\infty \frac{(\log x)^n}{(1+x)2}dx$$

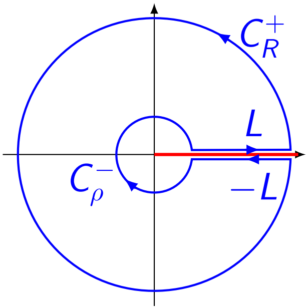


Thus

$$\oint_C \frac{(\log z)^N}{(1+z)^2}dz = 2\pi\mathrm{i}\times\left\{-N(\mathrm{i}\pi)^{N-1}\right\}$$

Integrands with logarithmic branch points

$$I_n = \int_0^\infty \frac{(\log x)^n}{(1+x)2} dx$$

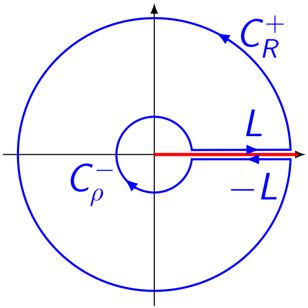


Thus

$$\oint_C \frac{(\log z)^N}{(1+z)^2} dz = -2N(\pi \mathfrak{i})^N$$

Integrands with logarithmic branch points

$$I_n = \int_0^\infty \frac{(\log x)^n}{(1+x)2} dx$$

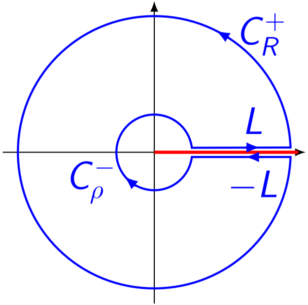


We have

$$\sum_{k=0}^{N-1} \binom{N}{k} (2\pi i)^{N-k} I_k = 2N(i\pi)^N$$

Integrands with logarithmic branch points

$$I_n = \int_0^\infty \frac{(\log x)^n}{(1+x)2} dx$$



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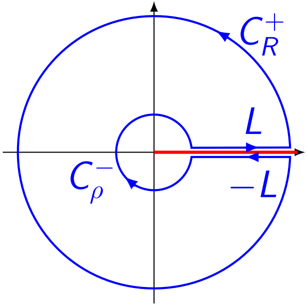
$$\sum_{k=0}^{N-1} \binom{N}{k} (2\pi \mathrm{i})^{N-k} I_k = 2N(\mathrm{i}\pi)^N$$

Putting $N = 1$, we get $I_0 = 1$

Integrands with logarithmic branch points

$$I_n = \int_0^\infty \frac{(\log x)^n}{(1+x)2} dx$$



We have

$$\sum_{k=0}^{N-1} \binom{N}{k} (2\pi \mathrm{i})^{N-k} I_k = 2N(\mathrm{i}\pi)^N$$

Putting $N = 1$, we get $I_0 = 1$

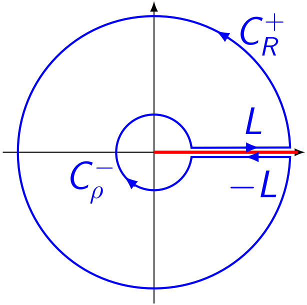
Putting $N = 2$,

$$(2\pi \mathrm{i})^2 I_0 + 2 \times (2\pi \mathrm{i}) I_1 = 4(\mathrm{i}\pi)^2$$

Integrands with logarithmic branch points

$$I_n = \int_0^\infty \frac{(\log x)^n}{(1+x)2} dx$$

We have



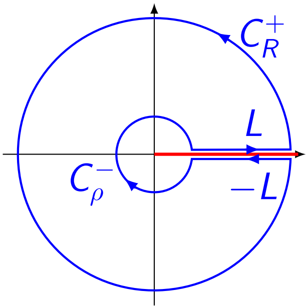
$$\sum_{k=0}^{N-1} \binom{N}{k} (2\pi i)^{N-k} I_k = 2N(i\pi)^N$$

Putting $N = 1$, we get $I_0 = 1$
Putting $N = 2$,

$$-4\pi^2 I_0 + 4\pi i I_1 = -4\pi^2$$

Integrands with logarithmic branch points

$$I_n = \int_0^\infty \frac{(\log x)^n}{(1+x)2} dx$$



We have

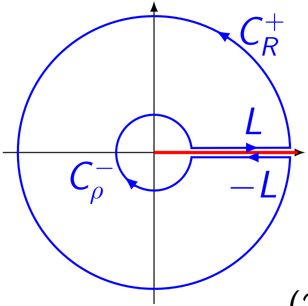
$$\sum_{k=0}^{N-1} \binom{N}{k} (2\pi \mathrm{i})^{N-k} I_k = 2N(\mathrm{i}\pi)^N$$

Putting $N = 1$, we get $I_0 = 1$

Putting $N = 2$, $\quad I_1 = 0$

Integrands with logarithmic branch points

$$I_n = \int_0^\infty \frac{(\log x)^n}{(1+x)2} dx$$



We have

$$\sum_{k=0}^{N-1} \binom{N}{k} (2\pi i)^{N-k} I_k = 2N(i\pi)^N$$

Putting $N = 1$, we get $I_0 = 1$

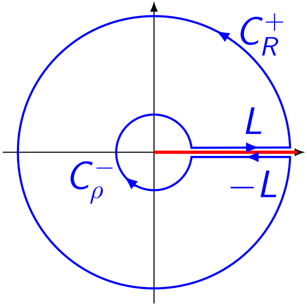
Putting $N = 2$, $\quad I_1 = 0$

Putting $N = 3$,

$$(2\pi i)^3 I_0 + 3 \times (2\pi i)^2 I_1 + 3 \times (2\pi i) I_2 = 6(i\pi)^3$$

Integrands with logarithmic branch points

$$I_n = \int_0^\infty \frac{(\log x)^n}{(1+x)2} dx$$



We have

$$\sum_{k=0}^{N-1} \binom{N}{k} (2\pi i)^{N-k} I_k = 2N(i\pi)^N$$

Putting $N = 1$, we get $I_0 = 1$

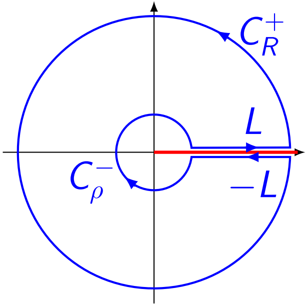
Putting $N = 2$, $\quad I_1 = 0$

Putting $N = 3$,

$$-8\pi^3 i I_0 - 12\pi^2 I_1 + 6\pi i I_2 = -6\pi^3 i$$

Integrands with logarithmic branch points

$$I_n = \int_0^\infty \frac{(\log x)^n}{(1+x)2} dx$$



We have

$$\sum_{k=0}^{N-1} \binom{N}{k} (2\pi \mathrm{i})^{N-k} I_k = 2N(\mathrm{i}\pi)^N$$

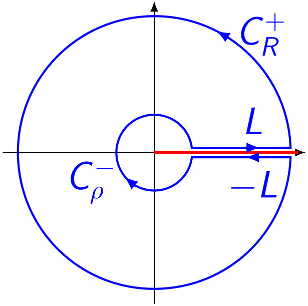
Putting $N = 1$, we get $I_0 = 1$

Putting $N = 2$, $I_1 = 0$

Putting $N = 3$, $I_2 = \frac{1}{3}\pi^2$

Integrands with logarithmic branch points

$$I_n = \int_0^\infty \frac{(\log x)^n}{(1+x)2} dx$$



We have

$$\sum_{k=0}^{N-1} \binom{N}{k} (2\pi \mathrm{i})^{N-k} I_k = 2N(\mathrm{i}\pi)^N$$

Putting $N = 1$, we get $I_0 = 1$

Putting $N = 2$, $\quad I_1 = 0$

Putting $N = 3$, $\quad I_2 = \frac{1}{3}\pi^2$

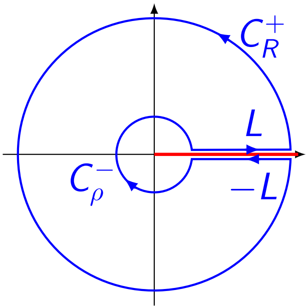
and so on ...

Integrands with logarithmic branch points

$$\int_0^\infty \frac{x}{1+x^3}dx$$

Integrands with logarithmic branch points

$$\int_0^\infty \frac{x}{1+x^3}\,dx$$



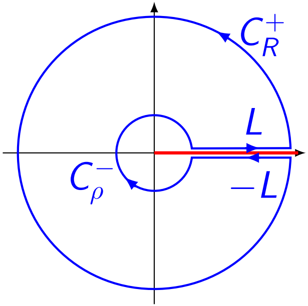
This can be calculated using

$$\oint_C \frac{z\log z}{1+z^3}\,dz$$

where C is the same contour as in the previous problem and we use the same branch of log.

Integrands with logarithmic branch points

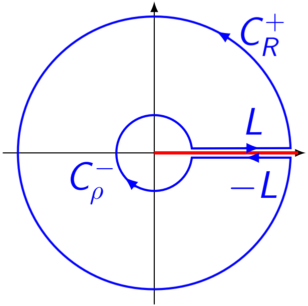
$$\int_0^\infty \frac{x}{1+x^3}dx$$



$$\oint_C \frac{z\log z}{1+z^3}dz = \left[\oint_{C_R^+} + \int_{-L} + \oint_{C_\rho^-} + \int_L\right]\frac{z\log z}{1+z^3}dz$$

Integrands with logarithmic branch points

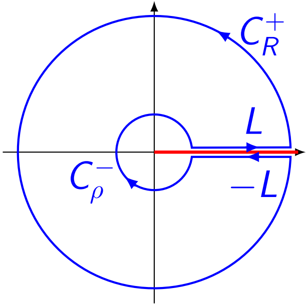
$$\int_0^\infty \frac{x}{1+x^3}dx$$



$$\lim_{\substack{R\to\infty\\ \rho\to 0}} \oint_C \frac{z\log z}{1+z^3}dz = \lim_{\substack{R\to\infty\\ \rho\to 0}} \left[\int_L + \int_{-L}\right] \frac{z\log z}{1+z^3}dz$$

Integrands with logarithmic branch points

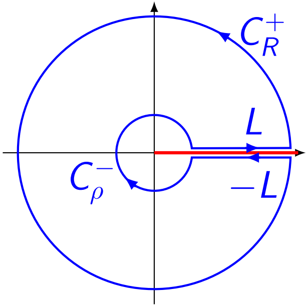
$$\int_0^\infty \frac{x}{1+x^3}dx$$



$$\lim_{\substack{R\to\infty\\ \rho\to 0}} \oint_C \frac{z\log z}{1+z^3}dz = \int_0^\infty \frac{x\log x}{1+x^3}dx + \int_\infty^0 \frac{x(\log x + 2\pi i)}{1+x^3}dx$$

Integrands with logarithmic branch points

$$\int_0^\infty \frac{x}{1+x^3}dx$$



$$\lim_{\substack{R\to\infty\\ \rho\to 0}} \oint_C \frac{z\log z}{1+z^3}dz = -2\pi\mathrm{i}\int_0^\infty \frac{x}{1+x^3}dx$$

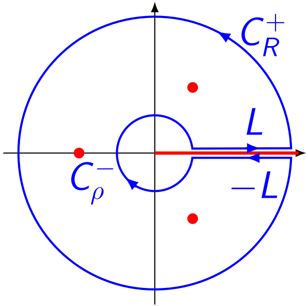
Integrands with logarithmic branch points

$$\int_0^\infty \frac{x}{1+x^3}dx$$

Integrands with logarithmic branch points

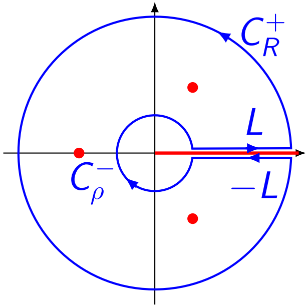
$$\int_0^\infty \frac{x}{1+x^3}\,dx$$

The integrand has three simple poles at $e^{i\pi/3}$, $e^{i\pi}$ and $e^{5i\pi/3}$, respectively.



Integrands with logarithmic branch points

$$\int_0^\infty \frac{x}{1+x^3}\,dx$$



The integrand has three simple poles at $e^{i\pi/3}$, $e^{i\pi}$ and $e^{5i\pi/3}$, respectively.
The residues are

$$\operatorname*{Res}_{z=e^{i\frac{p\pi}{3}}} \frac{z\log z}{1+z^3} = i\frac{p\pi}{9}e^{-i\frac{p\pi}{3}}, \quad p = 1, 3, 5$$

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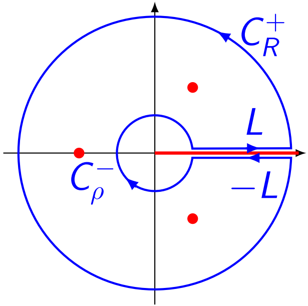

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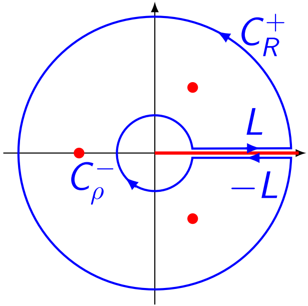

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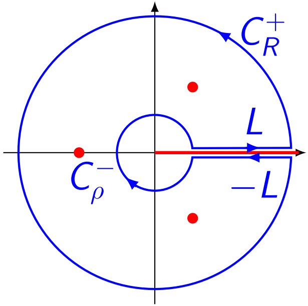

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Calculating $\int_0^\infty \frac{x}{1+x^3}dx$ by a different contour

We integrate

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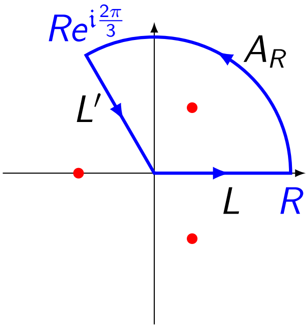

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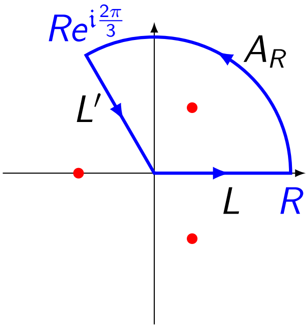

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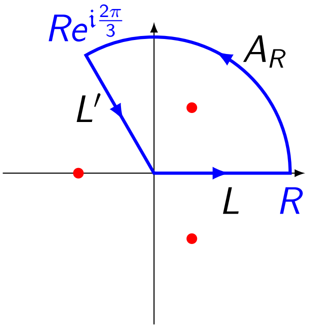

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$$\int_L \frac{z}{1+z^3}dz = \int_0^R \frac{x}{1+x^3}dx$$

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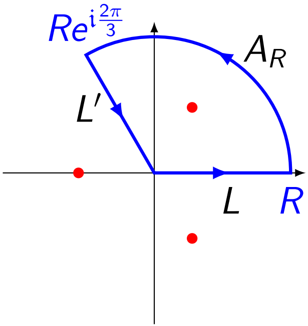

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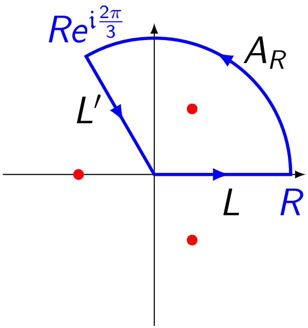

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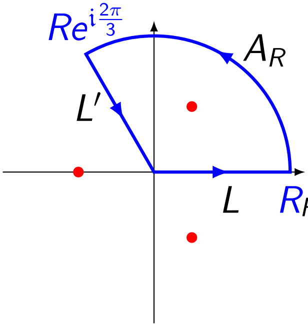

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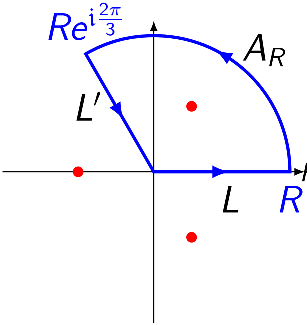

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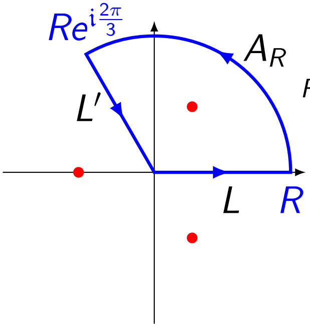

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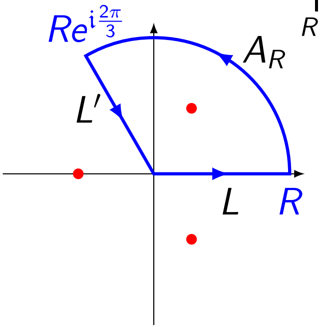

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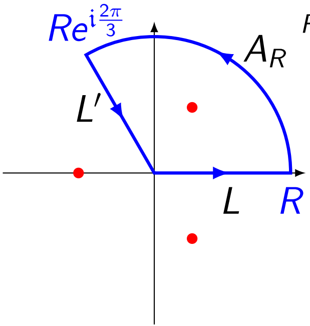

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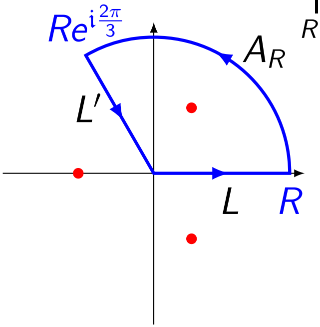

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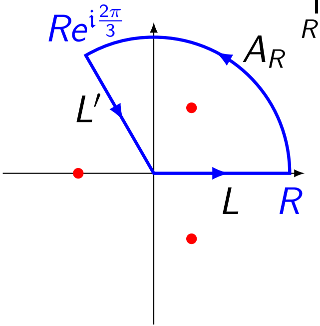

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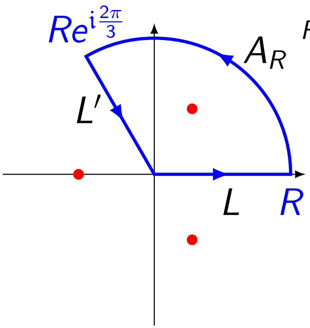

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Sums using the residue calculus

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The function

$$\cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)}$$

has simple poles at $0, \pm 1, \pm 2, \ldots$.

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Thus the function

$$\pi \cot(\pi z)$$

has a residue of 1 at *all integer values* of z.

Let f be a function that is analytic on $\mathbb{C}$ except for a finite set

$$E = \{z_1, z_2, \ldots, z_m\}$$

of isolated singularities. Also suppose that $\exists M, R > 0$:

$$|z| > R \implies |f(z)| \leq \frac{M}{|z|^a}, \; a > 1$$

Then

$$\sum_{n \in \mathbb{Z} \setminus E} f(n) = -\sum_{z_i \in E} \operatorname*{Res}_{z_i} \pi f(z) \cot(\pi z)$$

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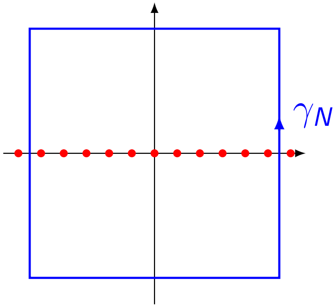
$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}}$$

Sums using the residue calculus

A special family of contours :



For each $N \in \mathbb{N}$ consider the contour γ_N

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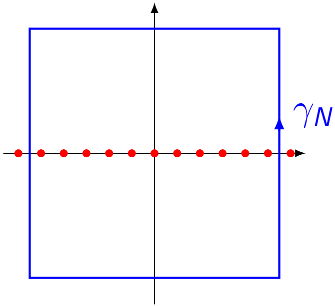

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A square with vertices at
$\left(N + \frac{1}{2}\right)(+1 + \mathrm{i})$,
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traversed in order.

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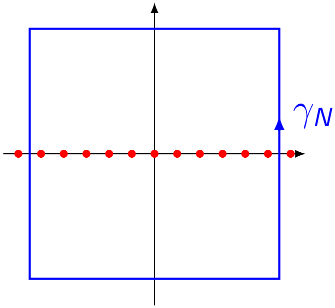

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- $\exists R > 0$:

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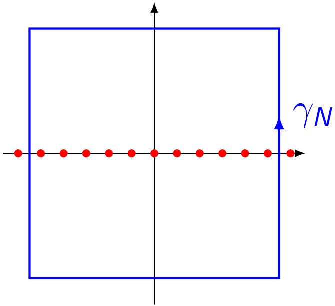

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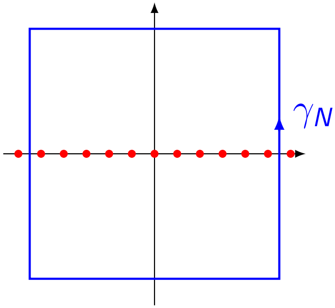

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- $\forall N \in \mathbb{N}$ we have ▸ Proof?

$$\int_{\gamma_N} \frac{\pi \cot(\pi z)}{z} dz = 0$$

Sums using the residue calculus

Proof of the summation formula

- $|z| \geq N + \frac{1}{2}$ on γ_N.

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- If $N > R$, $|f(z)| \leq \frac{M}{\left(N+\frac{1}{2}\right)^{\alpha}}$ on γ_N.

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- Length of the contour γ_N is $8\left(N + \frac{1}{2}\right)$.
- From the *ML* theorem,

$$\left| \oint_{\gamma_N} \pi \cot(\pi z) f(z) \right| \leq \frac{16\pi M}{\left(N + \frac{1}{2}\right)^{\alpha-1}}$$

Sums using the residue calculus

Proof of the summation formula

- $|z| \geq N + \frac{1}{2}$ on γ_N.
- If $N > R$, $|f(z)| \leq \frac{M}{\left(N+\frac{1}{2}\right)^{\alpha}}$ on γ_N.
- Thus $|\pi \cot(\pi z) f(z)| \leq \frac{2\pi M}{\left(N+\frac{1}{2}\right)^{\alpha}}$.
- Length of the contour γ_N is $8\left(N + \frac{1}{2}\right)$.
- From the ML theorem,

$$\left| \oint_{\gamma_N} \pi \cot(\pi z) f(z) \right| \leq \frac{16\pi M}{\left(N + \frac{1}{2}\right)^{\alpha - 1}}$$

- $\lim_{N \to \infty} \oint_{\gamma_N} \pi \cot(\pi z) f(z) = 0.$

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$\square$

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Go Back!

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