Complex Functions

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MA211, Lecture 4

Functions of a complex variable

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- The set D is called the domain of definition of the function f.
- The set of all images $R = \{w = f(z) \ : \ z \in D\}$ is called the range of f.

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$$\begin{aligned} u &= u(x, y) \\ v &= v(x, y) \end{aligned}$$

Examples

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$$\begin{aligned} u(x,y) &= x^4 - 6x^2y^2 + y^4 \\ v(x,y) &= 4x^3y - 4xy^3 \end{aligned}$$

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u(x, y) &= \exp(x)\cos(y) \\
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$$\begin{aligned} u(r,\theta) &= r^4 \cos(4\theta) + 4r^2 \cos(2\theta) - 6 \\ v(r,\theta) &= r^4 \sin(4\theta) + 4r^2 \sin(2\theta) \end{aligned}$$

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where

$$\begin{aligned} R &= R(r, \theta) \\ \Theta &= \Theta(r, \theta) \end{aligned}$$

“Double” polar version - examples :

$$f(z) = z^n, \qquad n \in \mathbb{Z}$$

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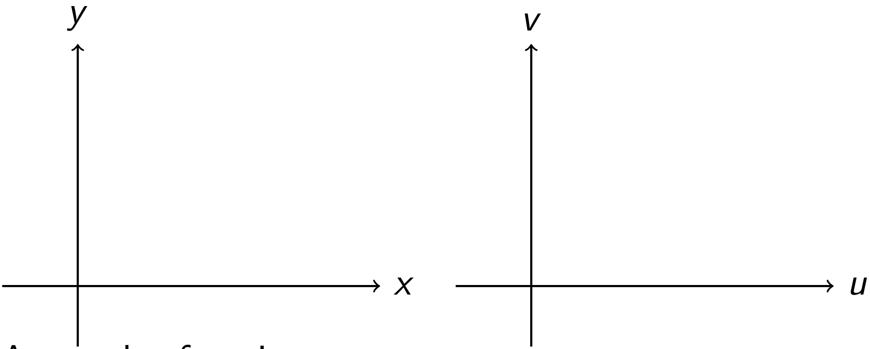
- But, then, the resulting Θs will also differ by some integer multiple of 2π.
- All's well with the world if $n \in \mathbb{Z}$!
- What if $n \notin \mathbb{Z}$?

Geometric interpretation of complex functions



A complex function

$$f \ : \ D \subset \mathbb{C} \to \mathbb{C}, \qquad w = f(z) = u(x,y) + \mathrm{i}v(x,y)$$

can be viewed as a mapping from D in the xy plane into the uv plane.

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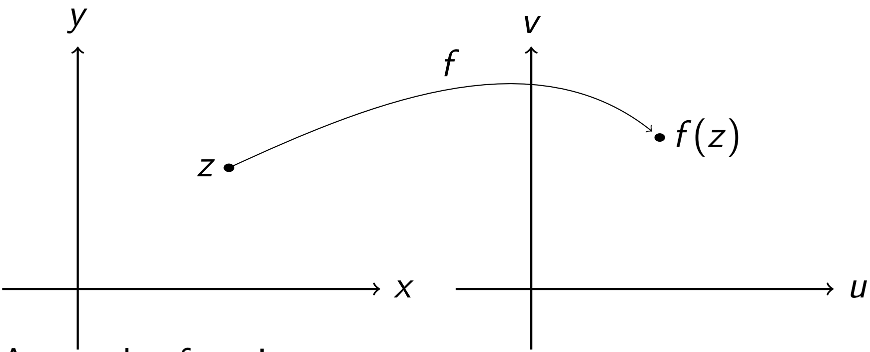

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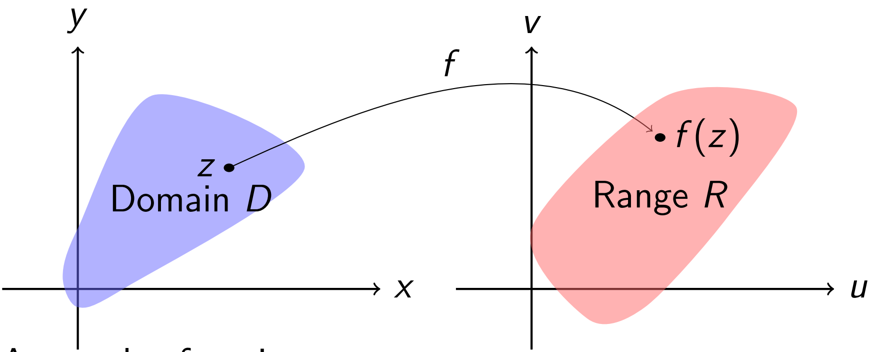

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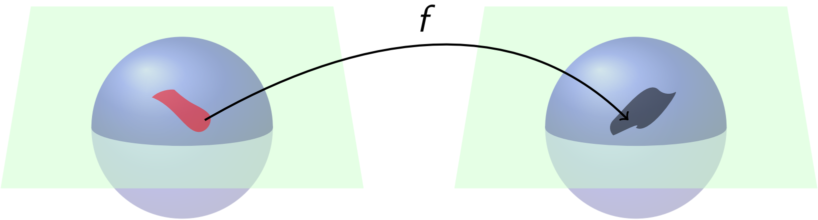
Complex functions on the Riemann sphere



The map $f : \mathbb{C} \to \mathbb{C}$ can also be represented as a map from a subset of the Riemann sphere into the Riemann sphere.

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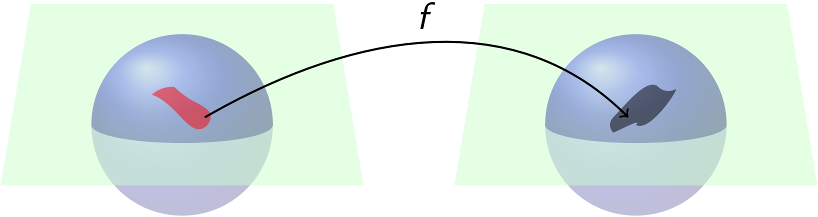


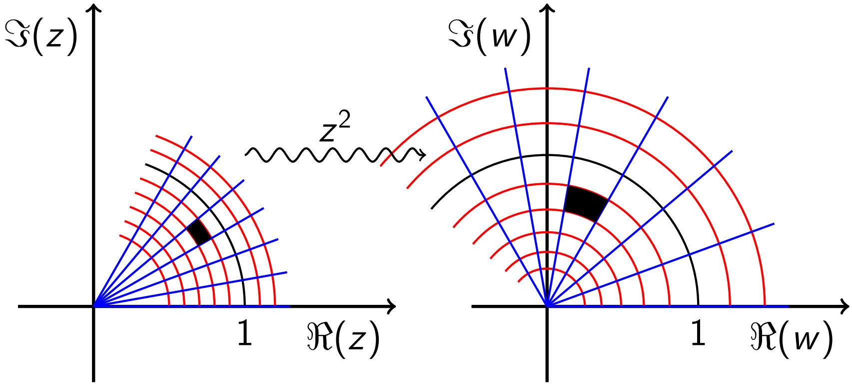
The map $f : \mathbb{C} \to \mathbb{C}$ can also be represented as a map from a subset of the Riemann sphere into the Riemann sphere.

This allows us to also include maps from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$, where $\mathbb{C}_\infty = C \cup \{\infty\}$ is the extended complex plane.

$f(z) = z^2$

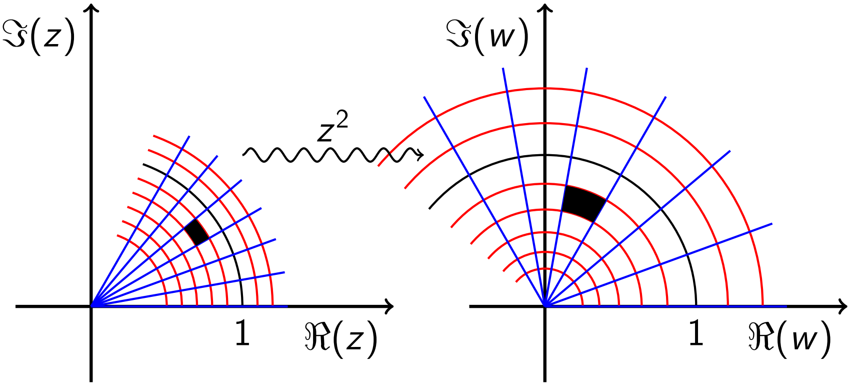
It is easy to understand the geometry in the polar representation :

$$z = re^{i\theta} \mapsto w = r^2 e^{2i\theta}$$



$f(z) = z^2$

- Straight lines through the origin transform into straight lines through the origin.
- Circles centered at the origin transform to circles centered on the origin.



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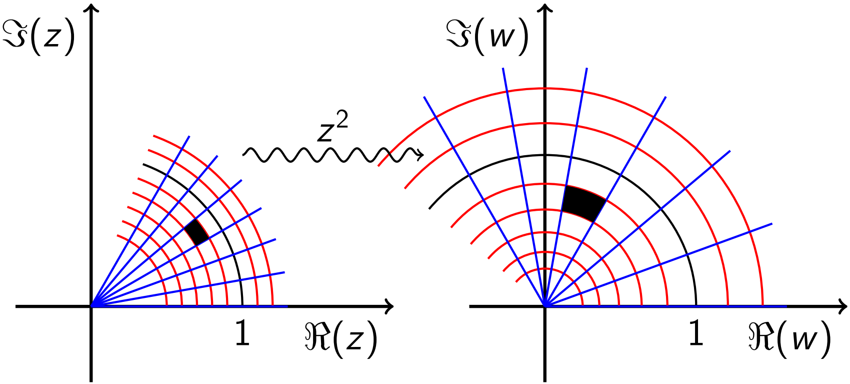
- The unit circle transforms into itself.
- Infinitesimal squares transform to infinitesimal squares.



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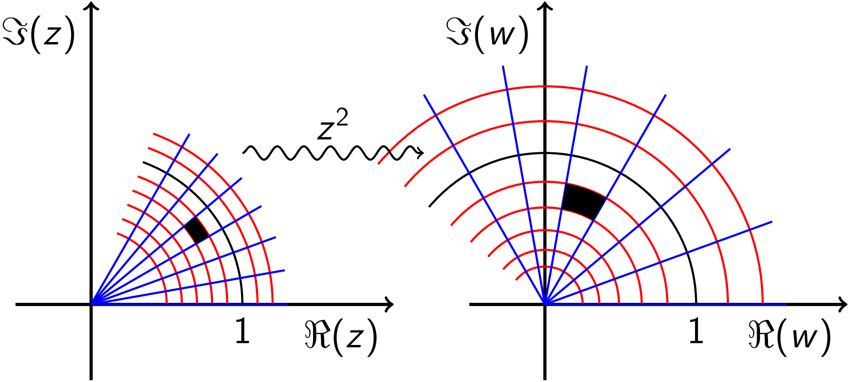
- Angles are preserved.
- Except at the origin, where they are doubled!



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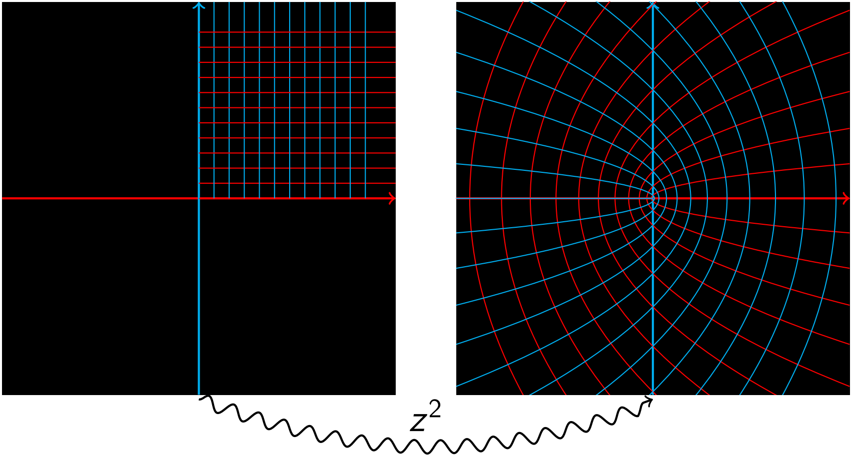
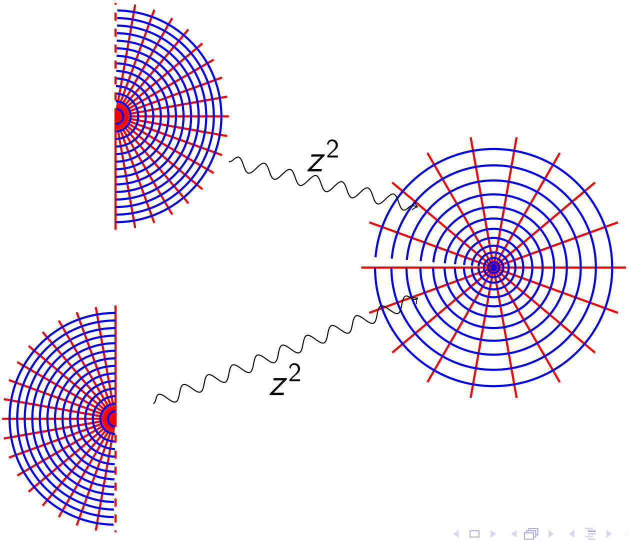
- Lines of constant x and y map into parabolas.
- The parabolas intersect at right angles!



$f(z) = z^2$

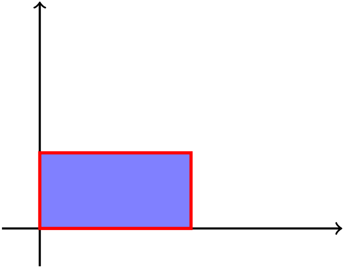
- Half the plane maps into the whole plane.
- The full plane maps onto two copies of itself!



Linear maps

The most general linear map is of the form

$$z \mapsto w = \alpha z + \beta$$



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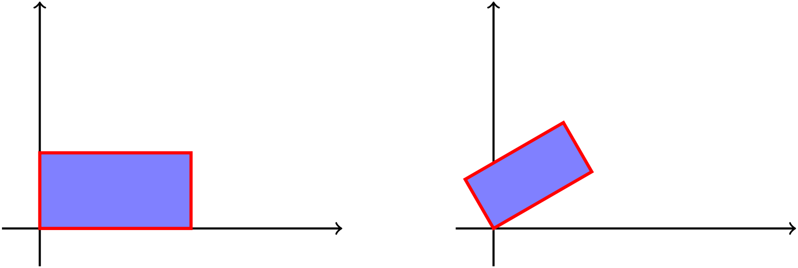

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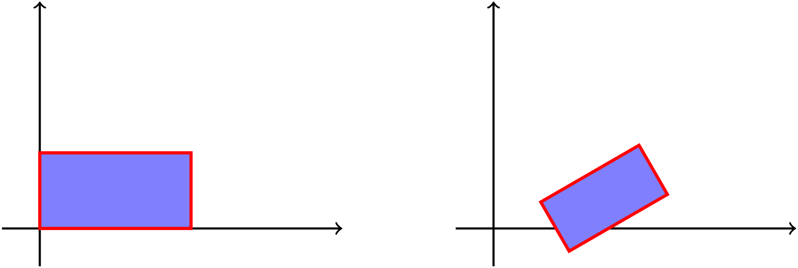

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None of these change angles between straight lines!