Complex Functions contd.

Ananda Dasgupta

MA211, Lecture 5

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"Double" polar form :

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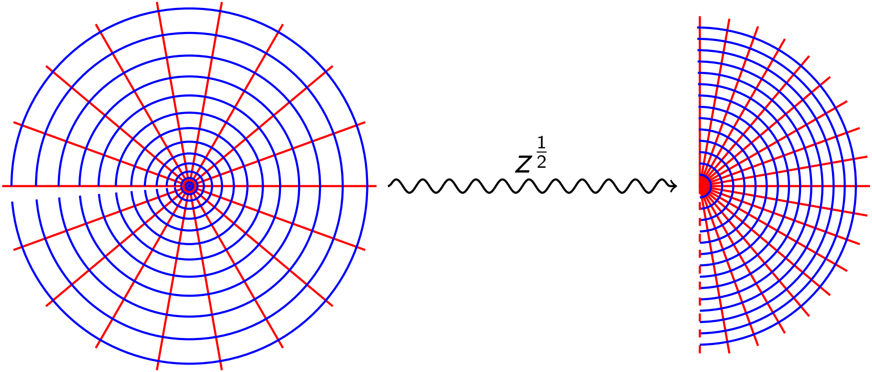
We restrict the domain D for f to $r > 0$, $-\pi < \theta \leq \pi$.

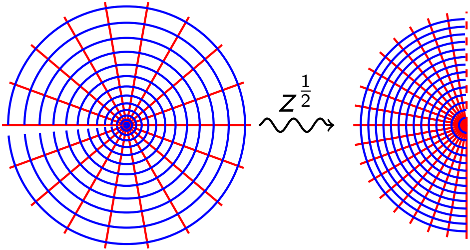
The image of the whole complex plane is the region $R > 0$, $-\frac{\pi}{2} < \Theta \leq \frac{\pi}{2}$ - the right half plane.

The function $z^{\frac{1}{2}}$, $\arg(z) \in (-\pi, \pi]$

- The complex plane maps into a half plane.



$z^{\frac{1}{2}}$ - the other branch : $\arg(z) \in (\pi, 3\pi]$



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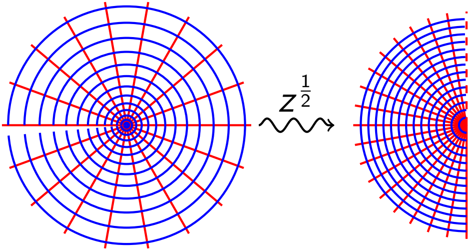

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- Then the whole complex plane maps onto the left half plane!

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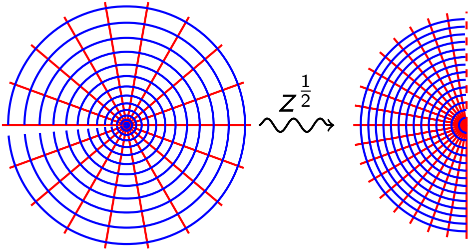

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- Then the whole complex plane maps onto the left half plane!
- The same point $re^{\mathrm{i}\theta} = re^{\mathrm{i}(\theta+2\pi)}$ maps to two different points $r^{\frac{1}{2}}e^{\mathrm{i}\frac{\theta}{2}}$ and $r^{\frac{1}{2}}e^{\mathrm{i}\left(\frac{\theta}{2}+\pi\right)} = -r^{\frac{1}{2}}e^{i\frac{\theta}{2}}$!

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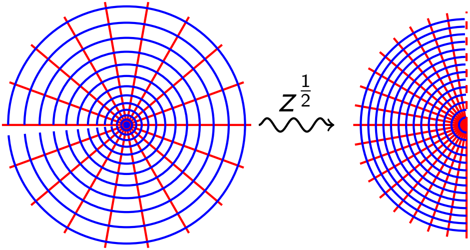

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- $z \mapsto z^{\frac{1}{2}}$ is **multi-valued**!

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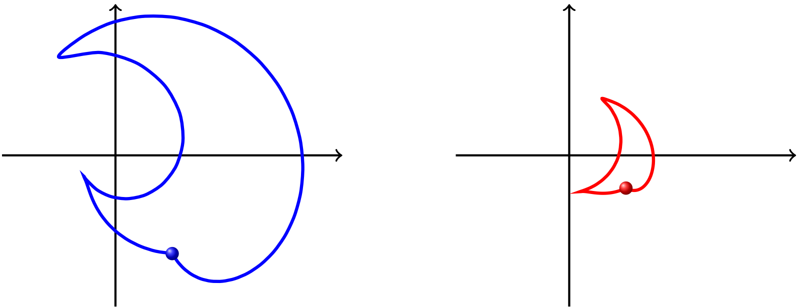
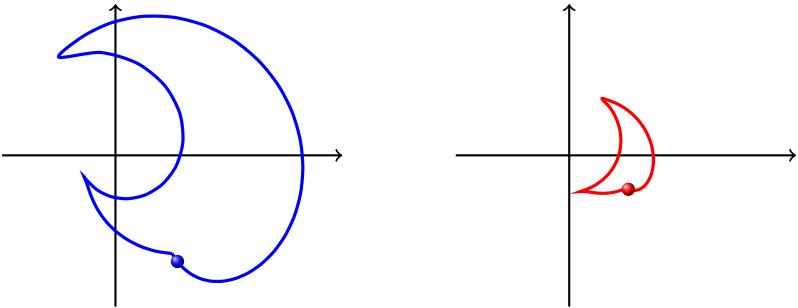
- Such functions are called **multifunctions**.

The multifunction $z \mapsto z^{\frac{1}{2}}$



As the point z traverses a closed curve, the point $z^{\frac{1}{2}}$ traverses another curve at half the angular velocity.

The multifunction $z \mapsto z^{\frac{1}{2}}$



If the closed curve returns to z_0 without going around the origin, the image returns to its original value.

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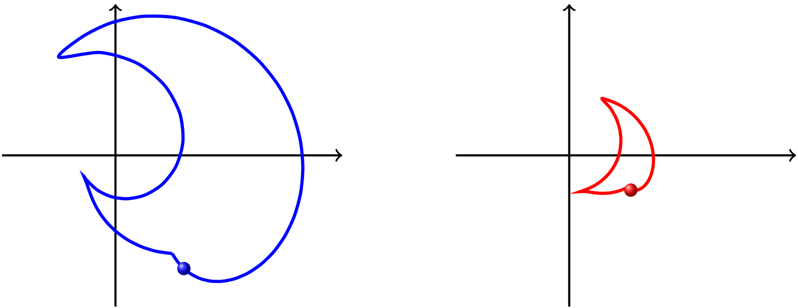

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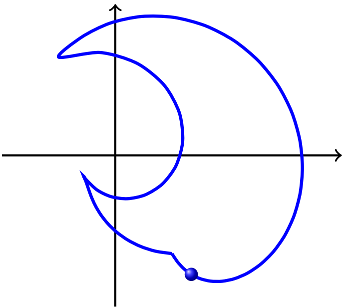

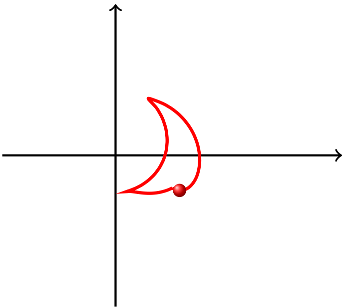

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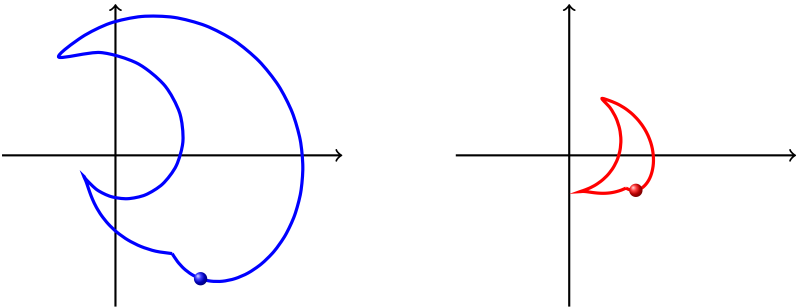

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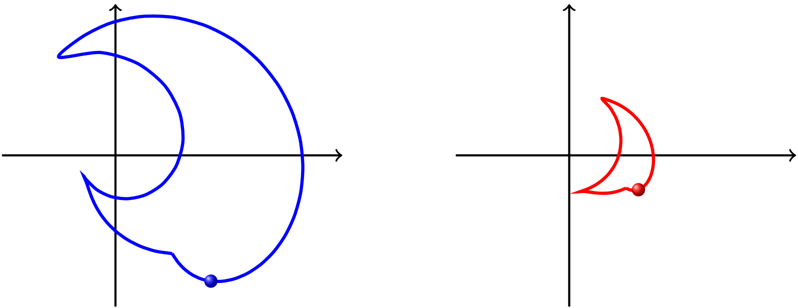

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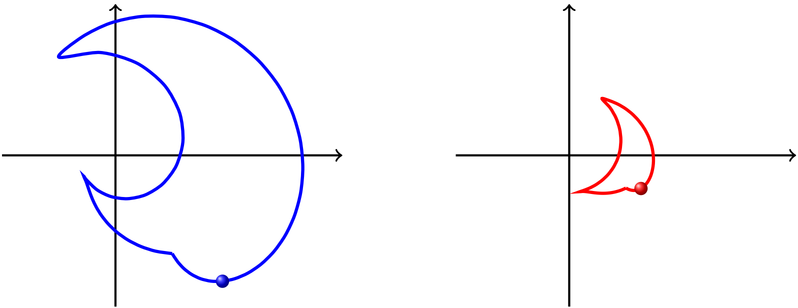

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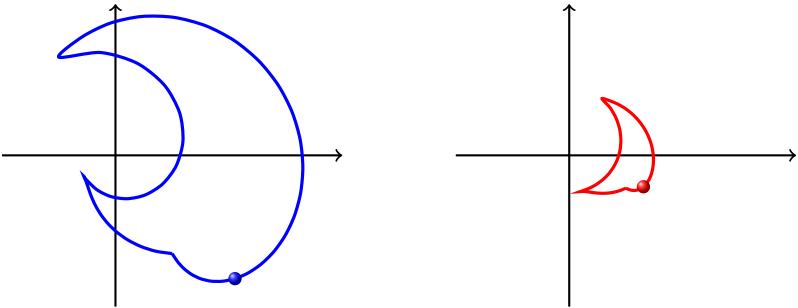

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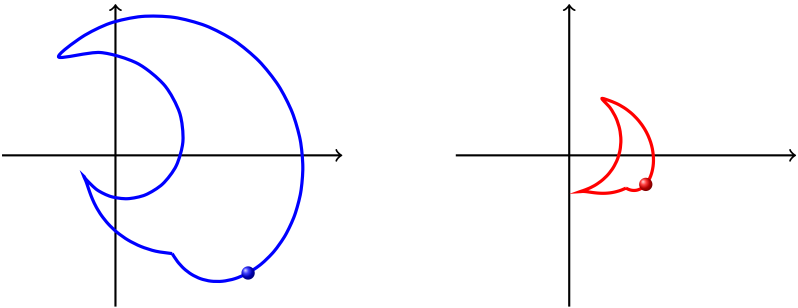

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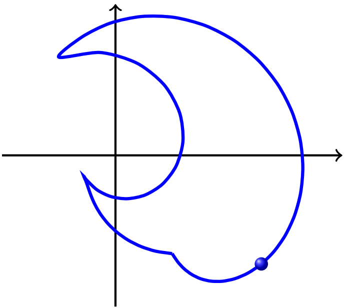

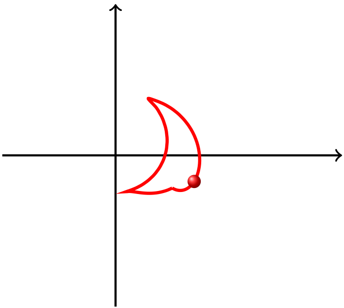

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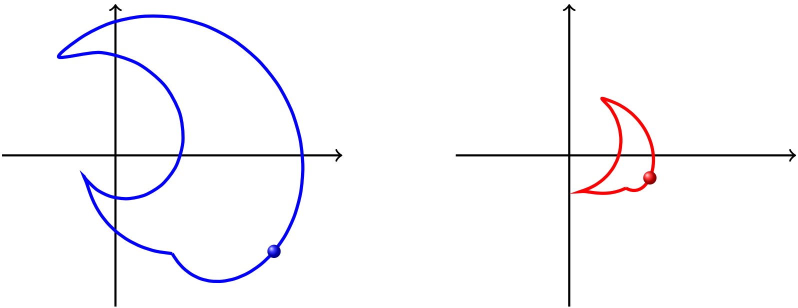

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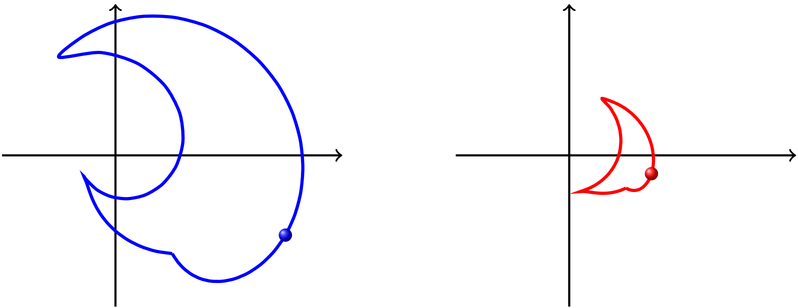

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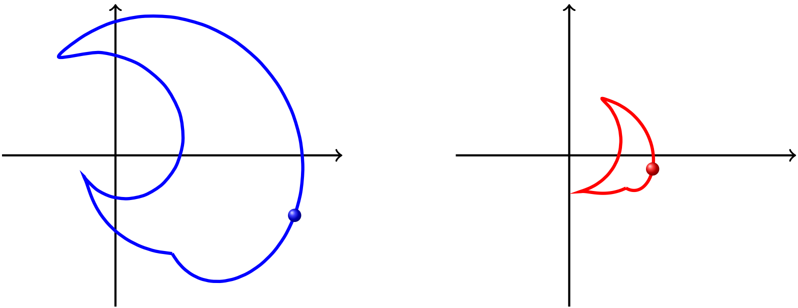

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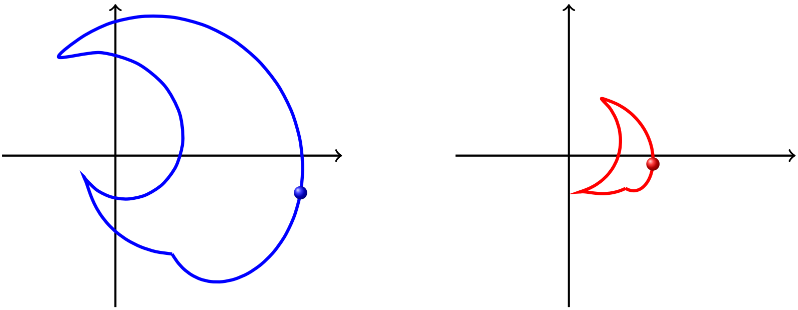

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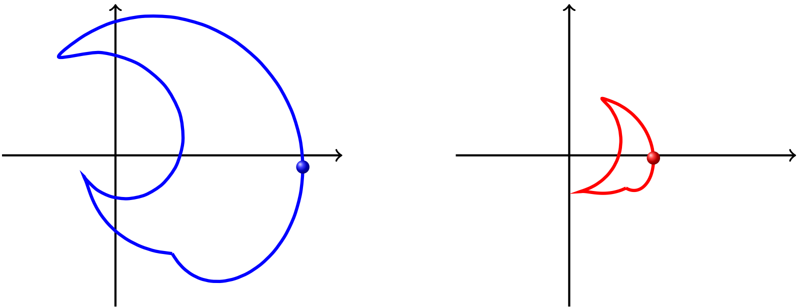

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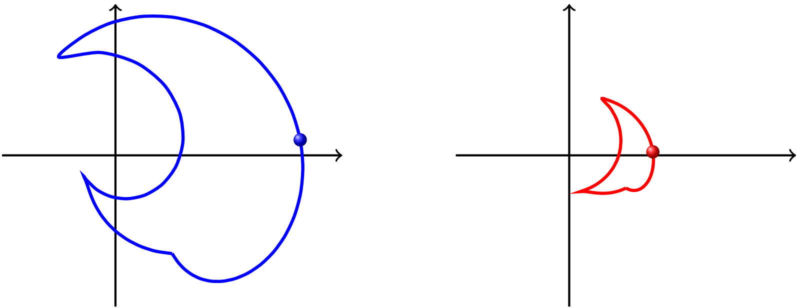

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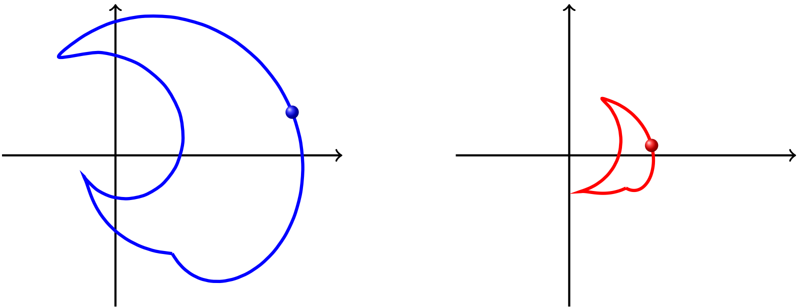

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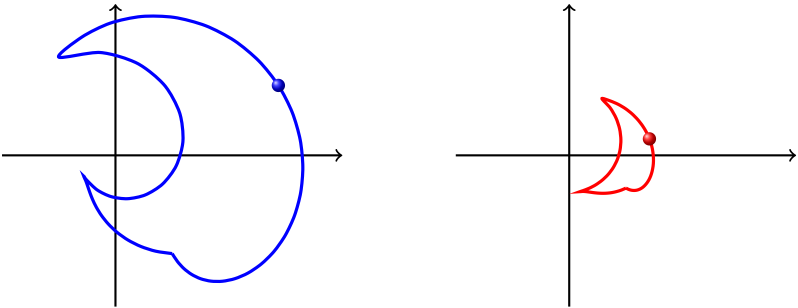

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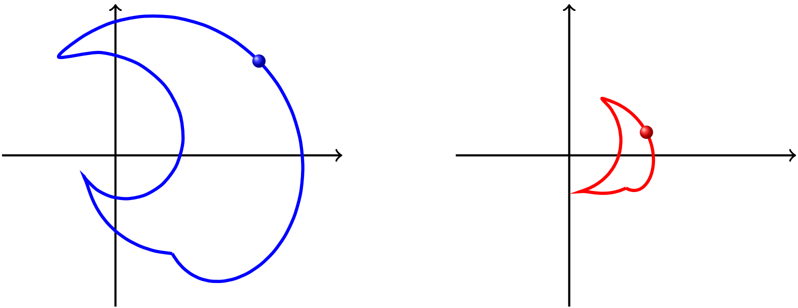

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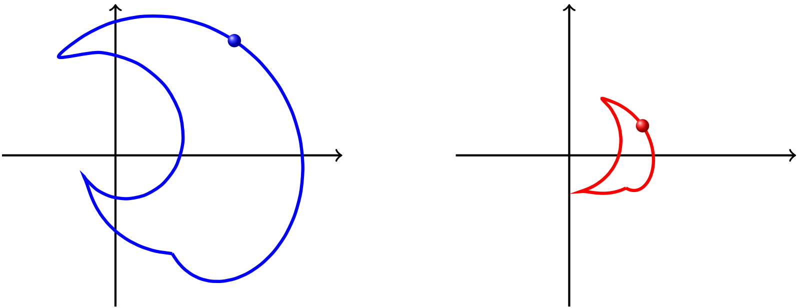

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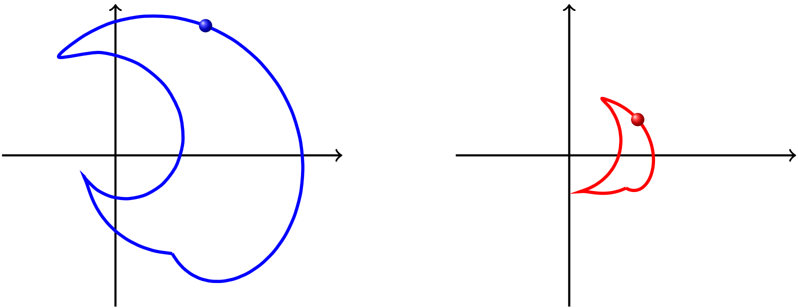

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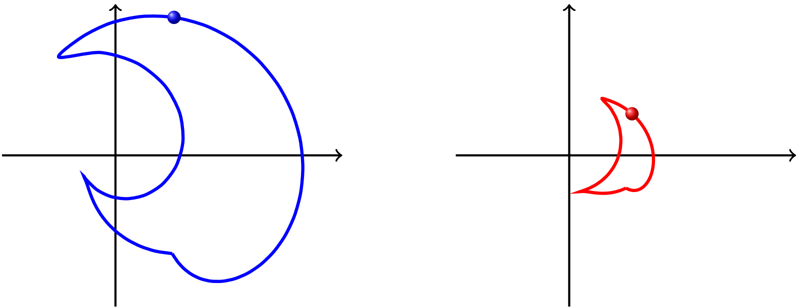

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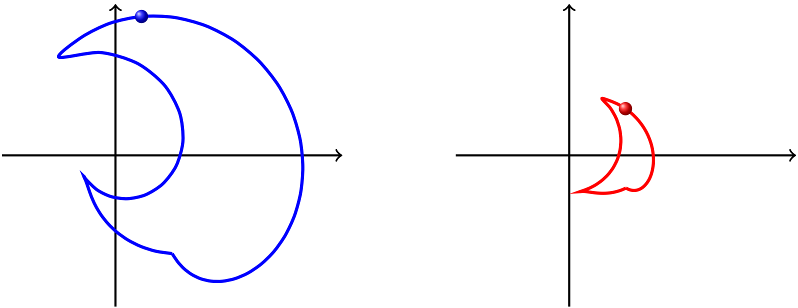

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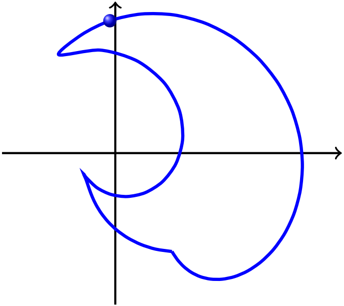

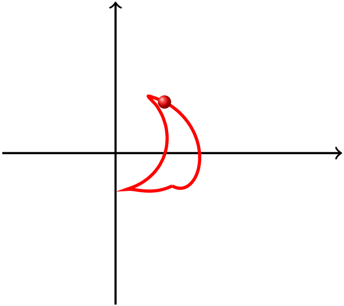

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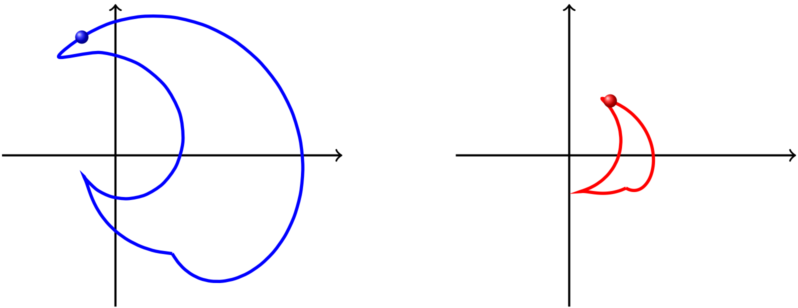

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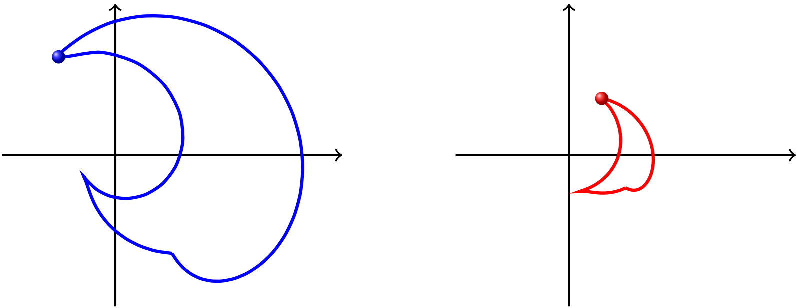

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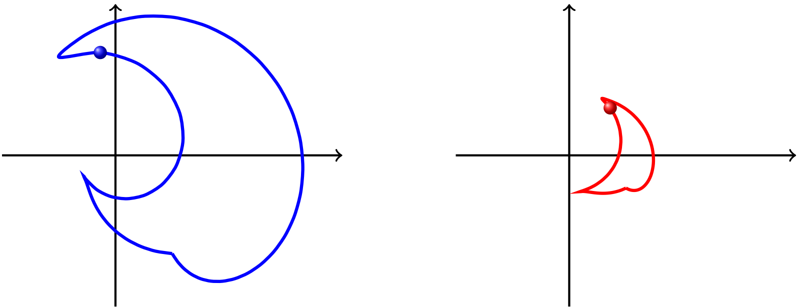

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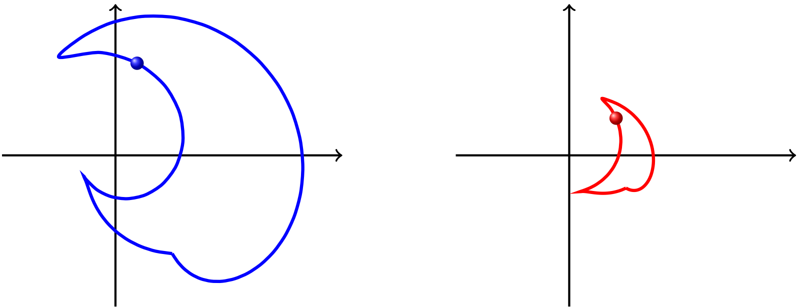

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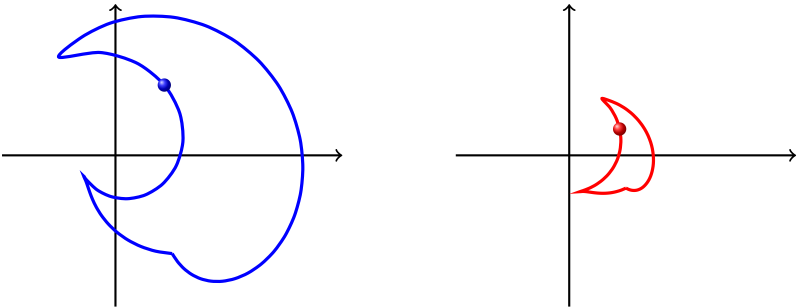

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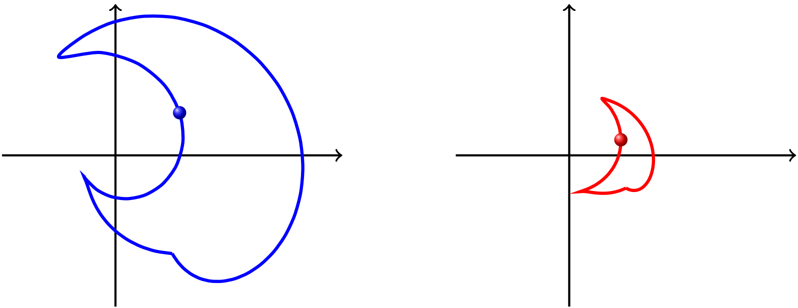

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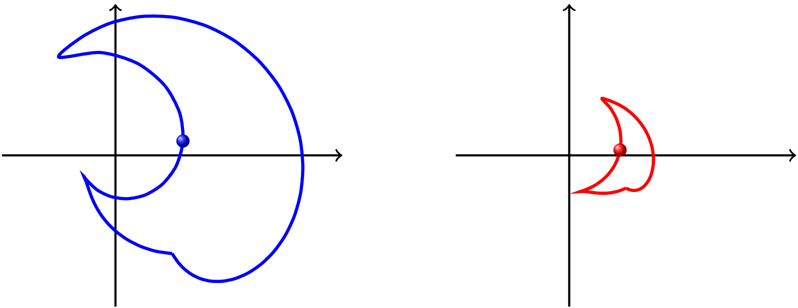

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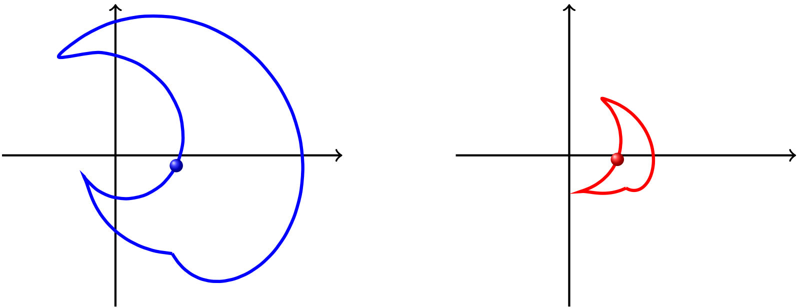

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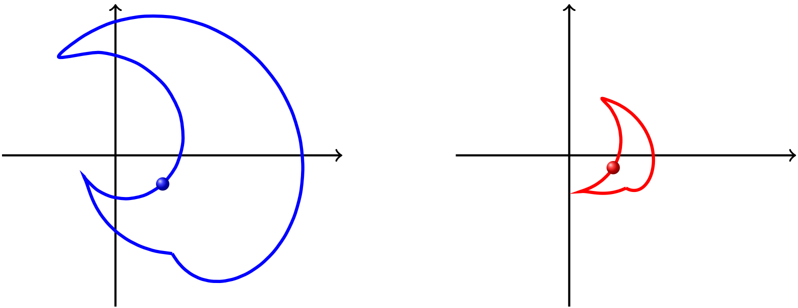

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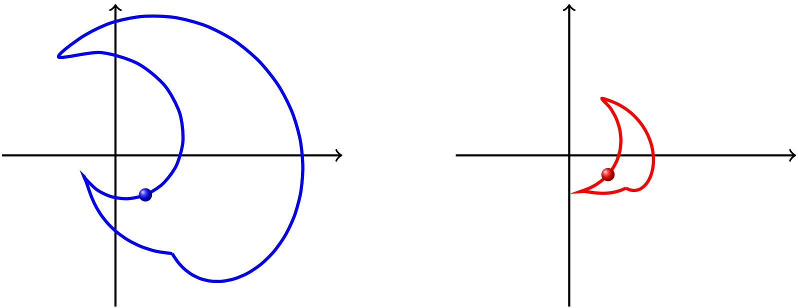

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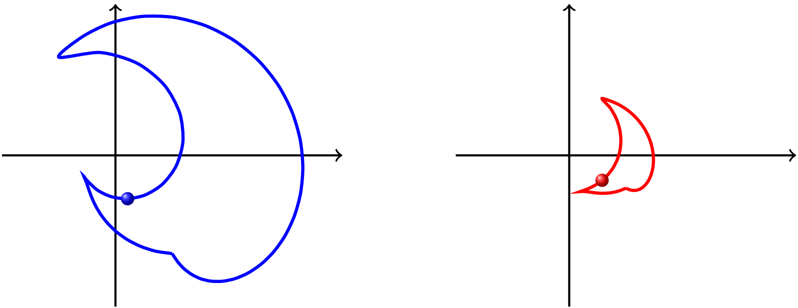

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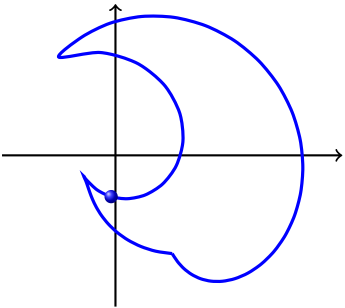

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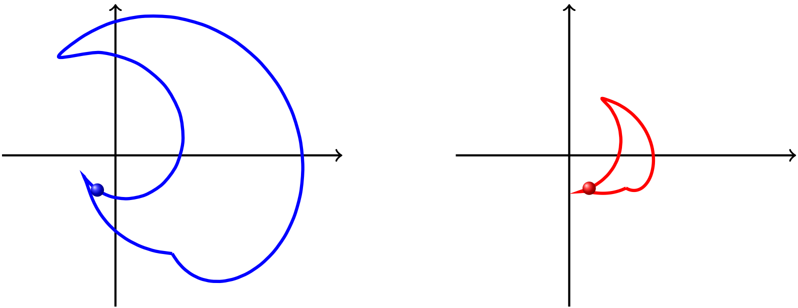

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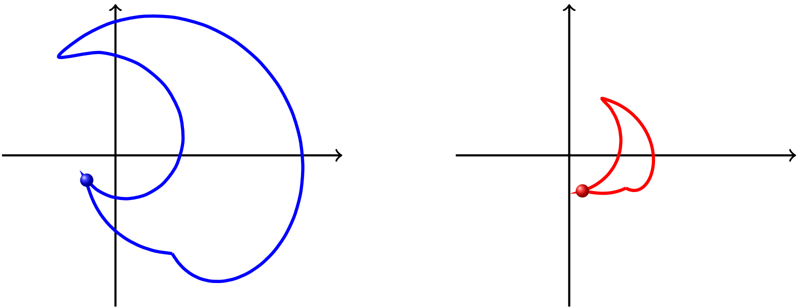

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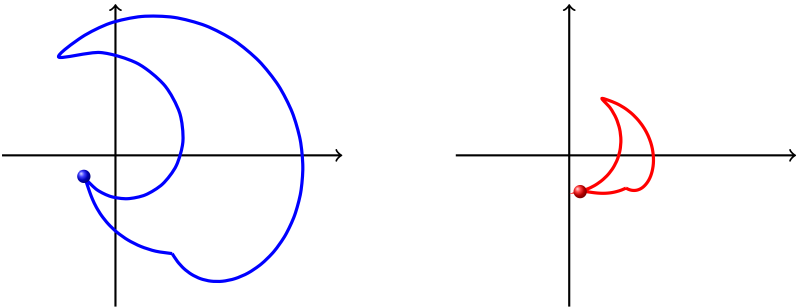

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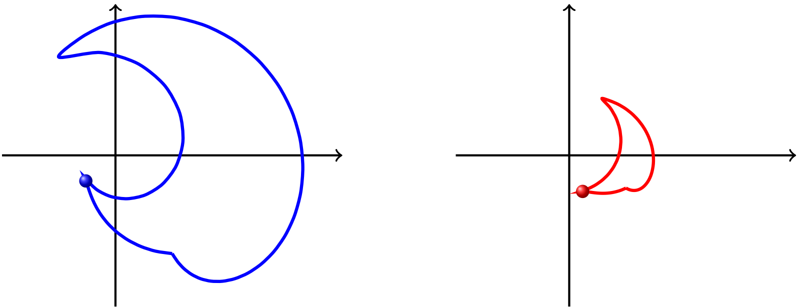

If the closed curve returns to z_0 without going around the origin, the image returns to its original value.

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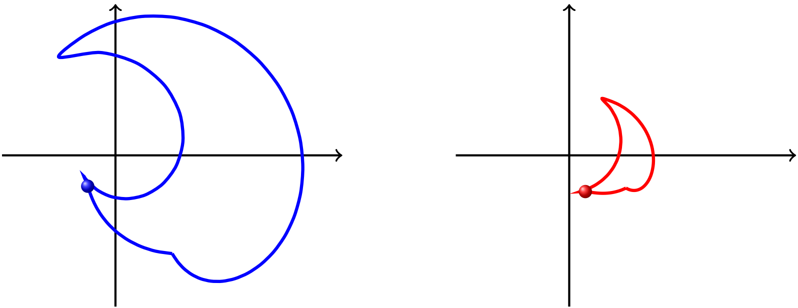

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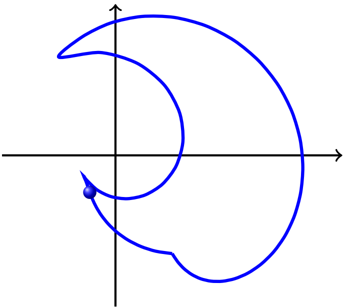

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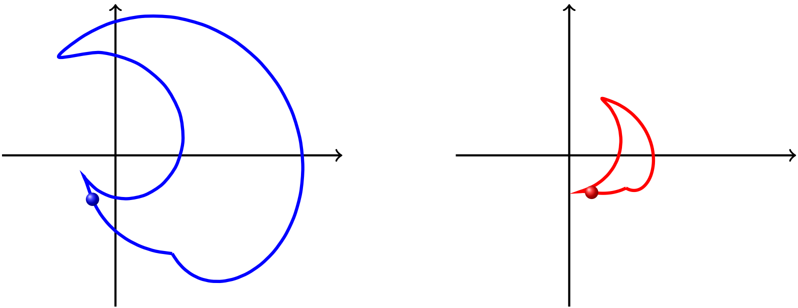

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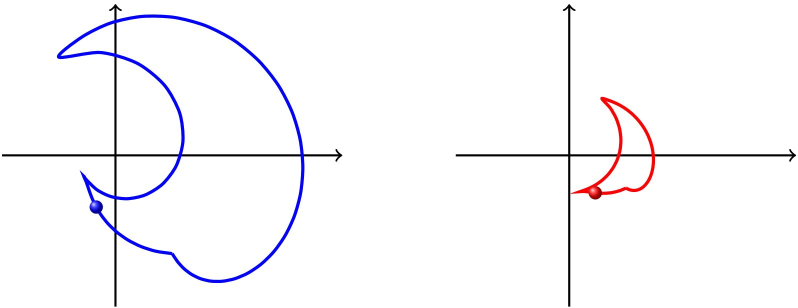

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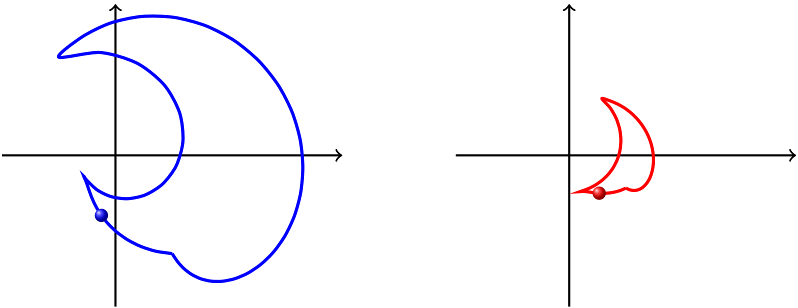

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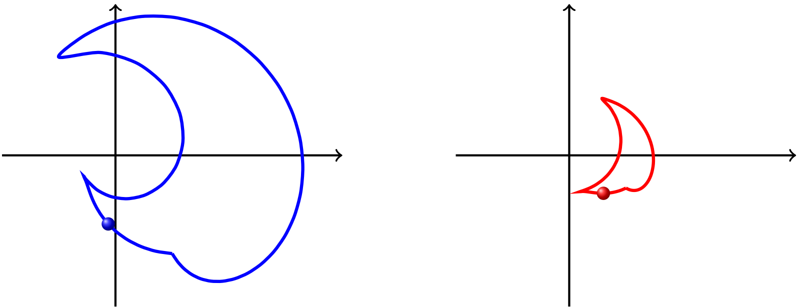

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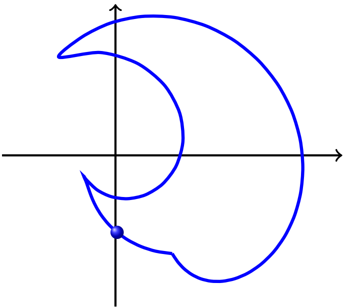

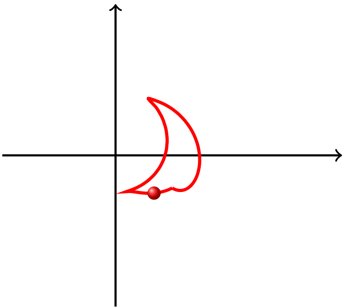

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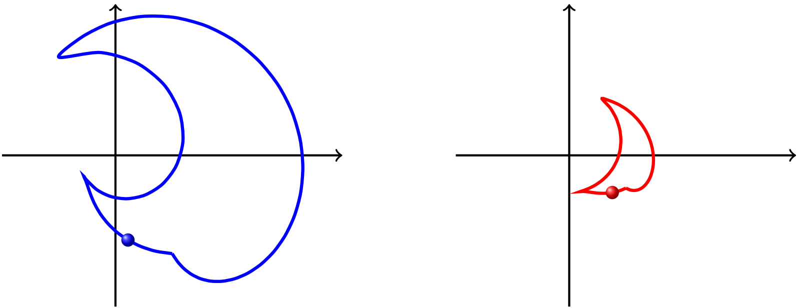

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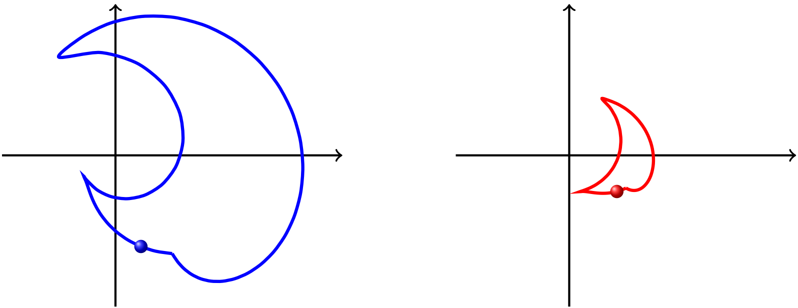

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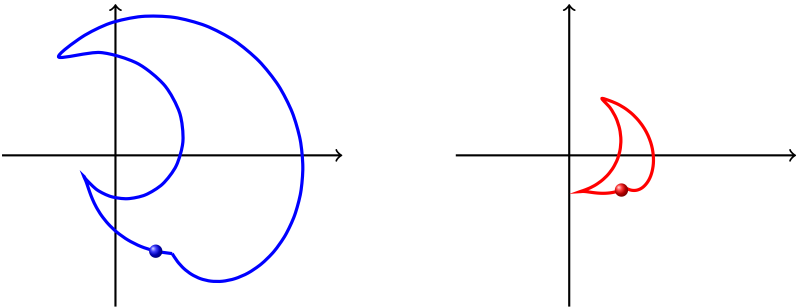

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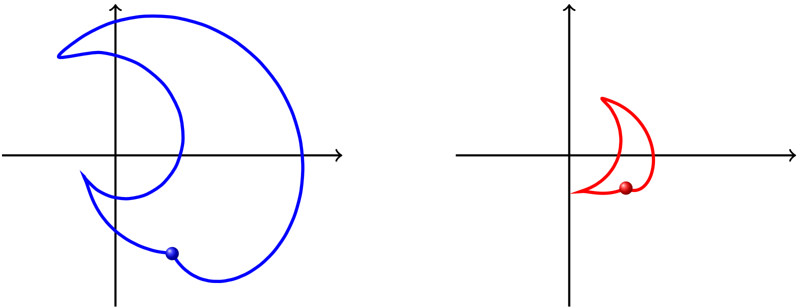

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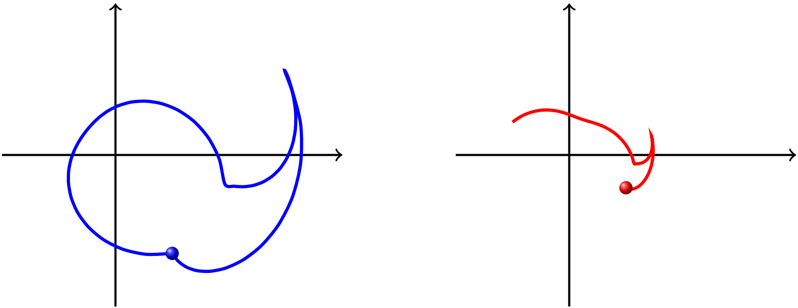

If the closed path encloses the origin, winding round it once, the value of $\arg(z)$ changes by 2π, and thus that of $\arg\left(z^{\frac{1}{2}}\right)$ changes by π.

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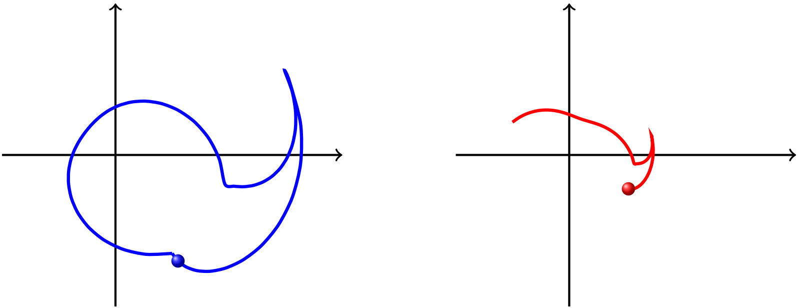

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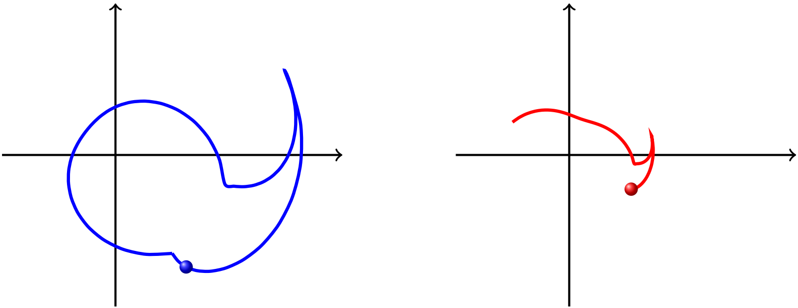

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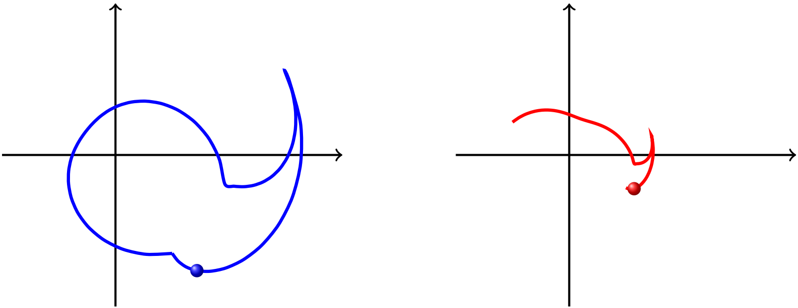

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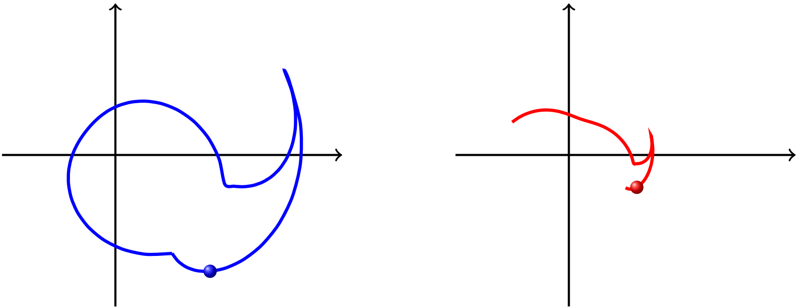

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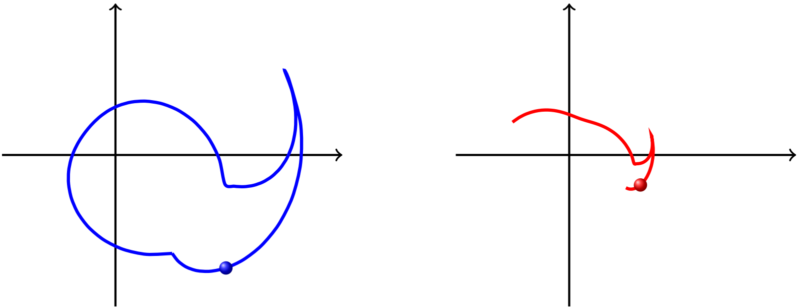

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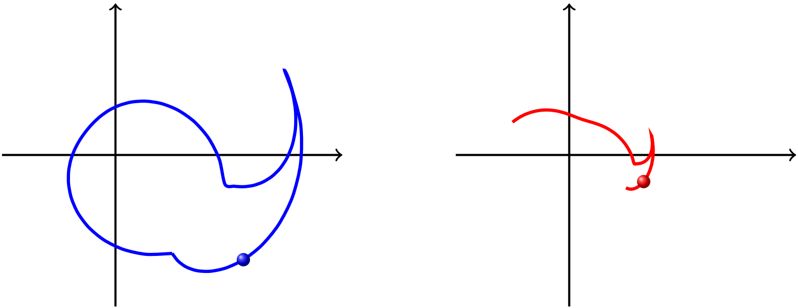

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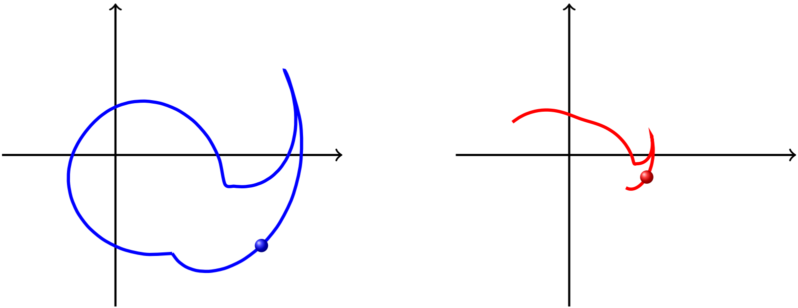

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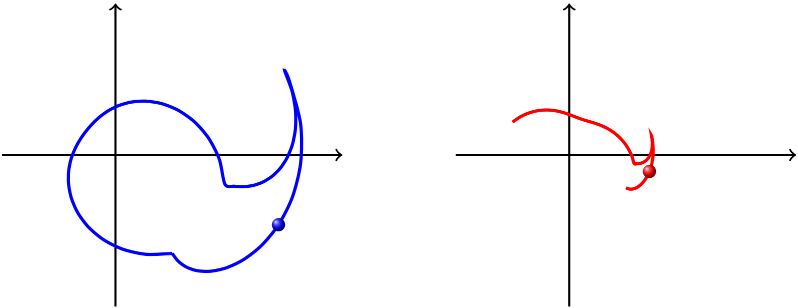

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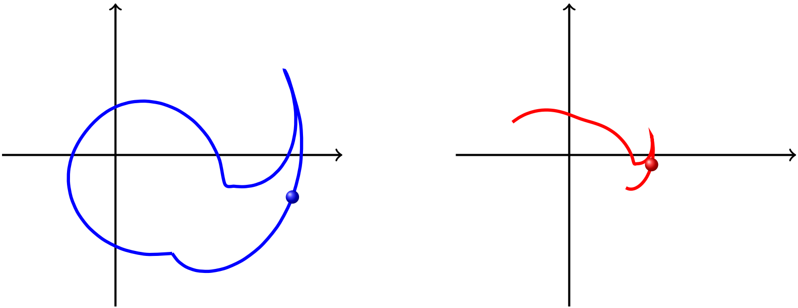

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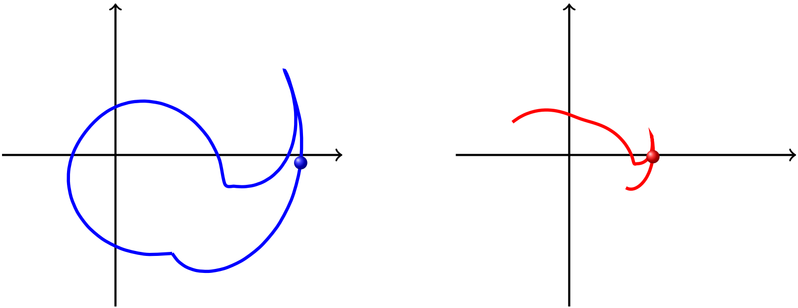

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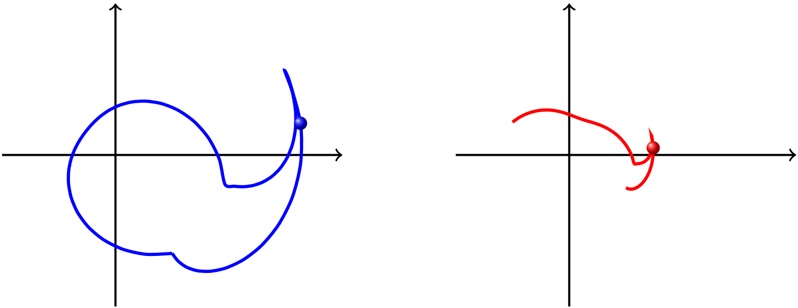

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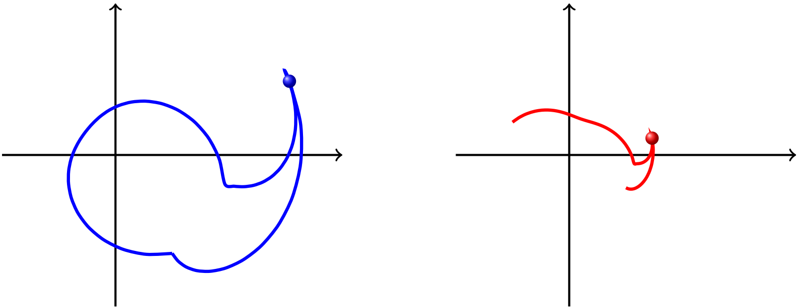

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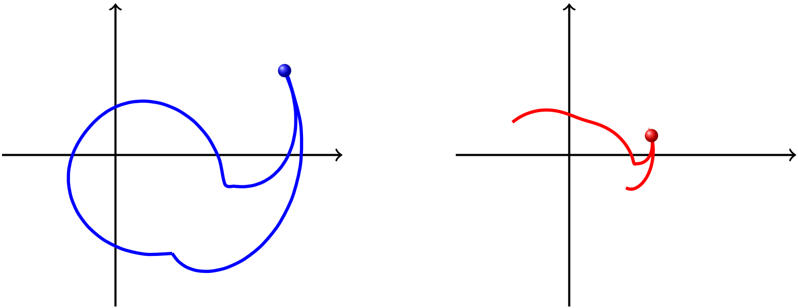

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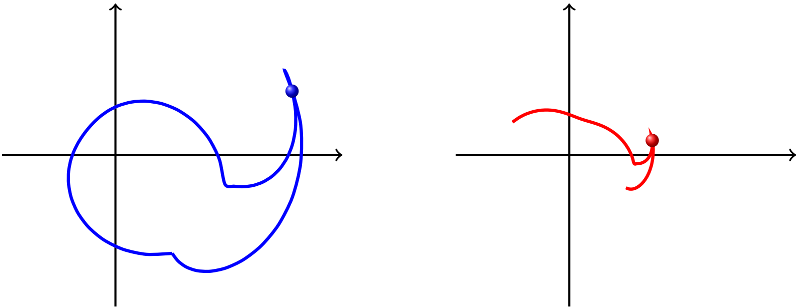

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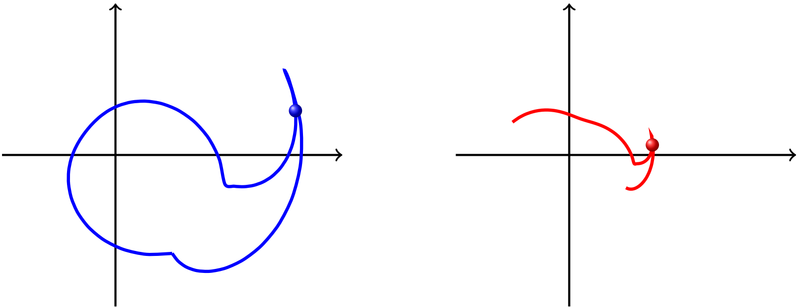

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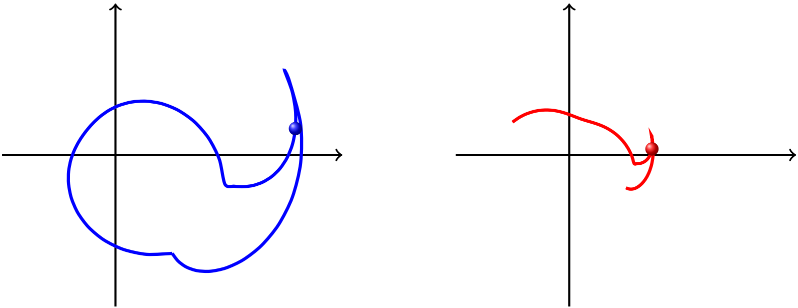

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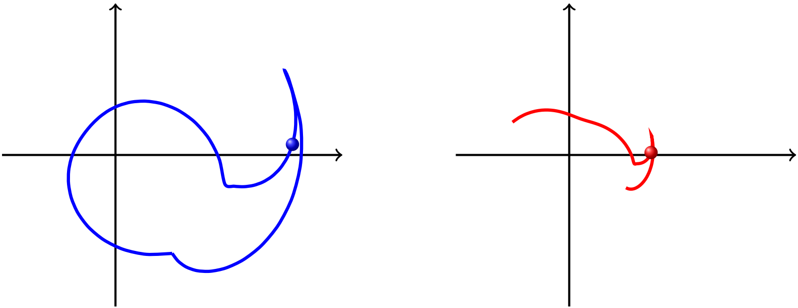

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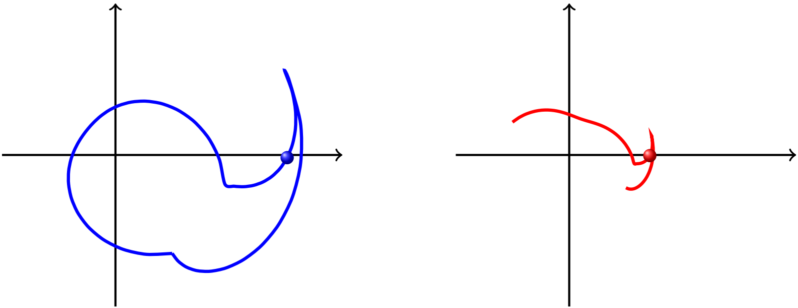

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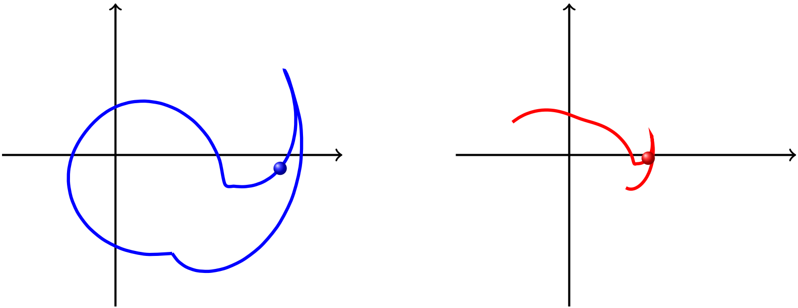

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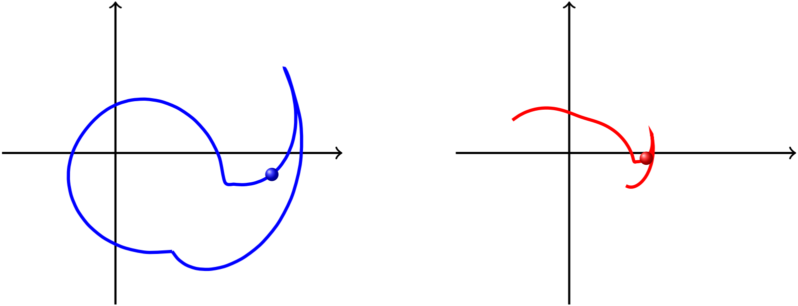

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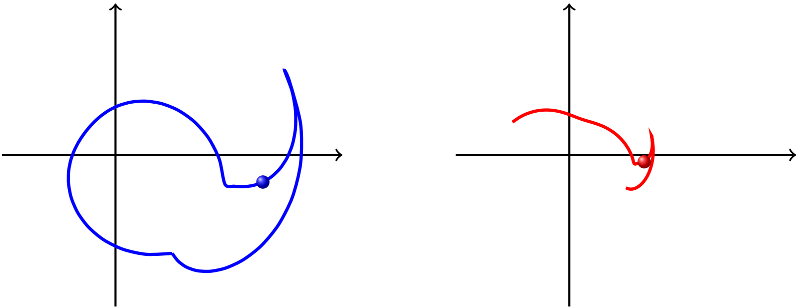

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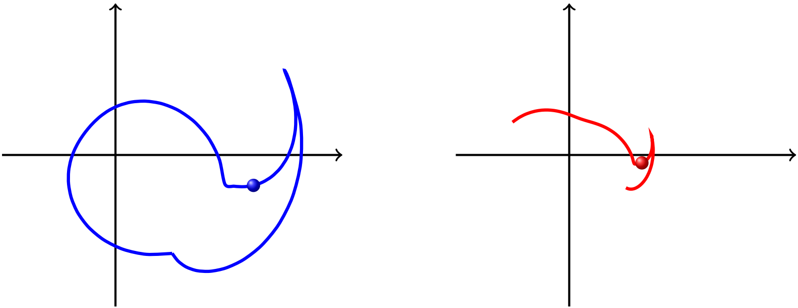

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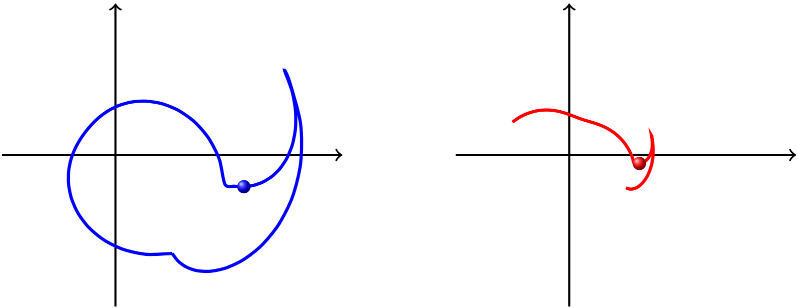

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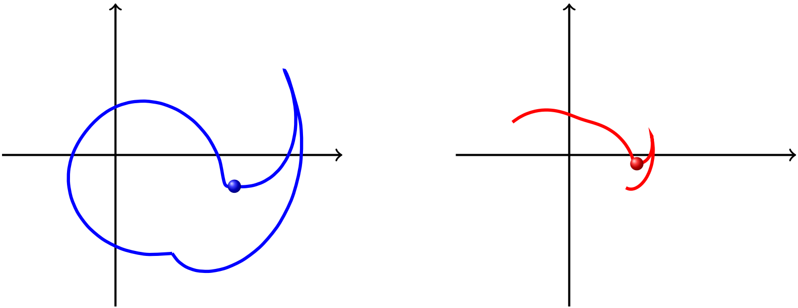

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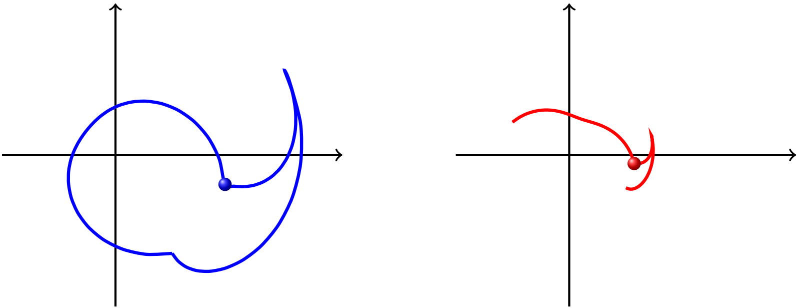

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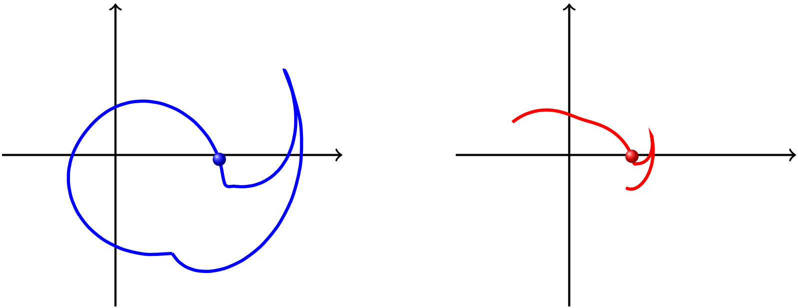

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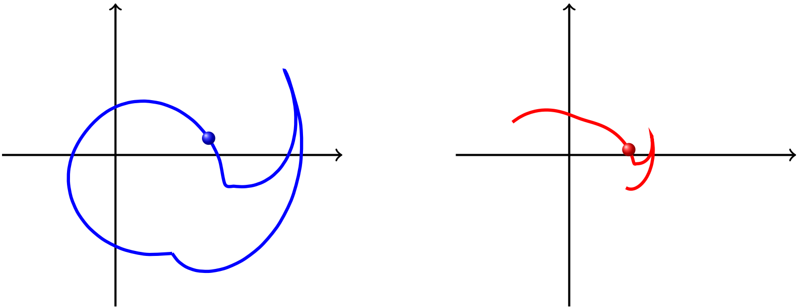

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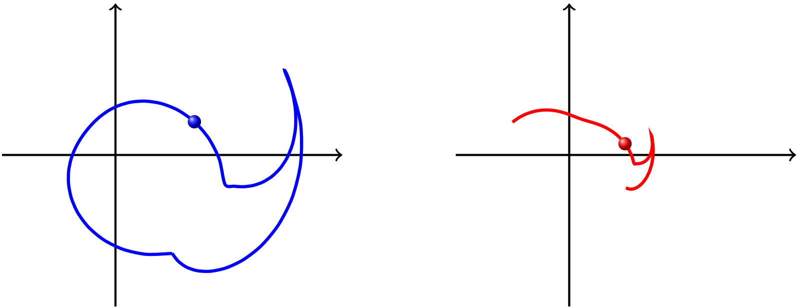

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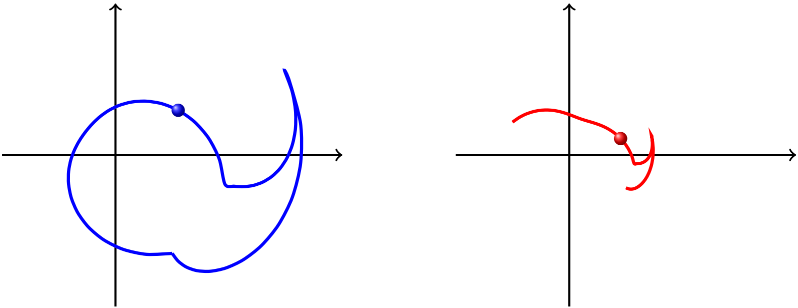

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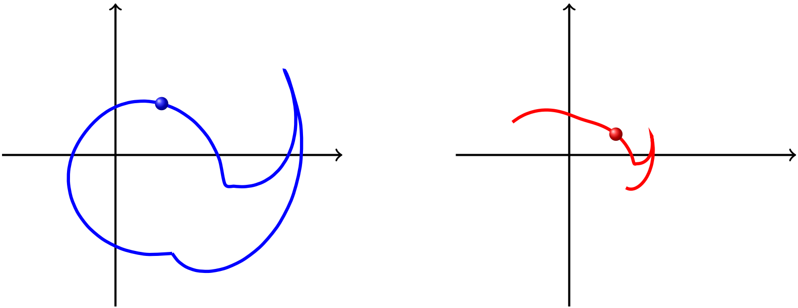

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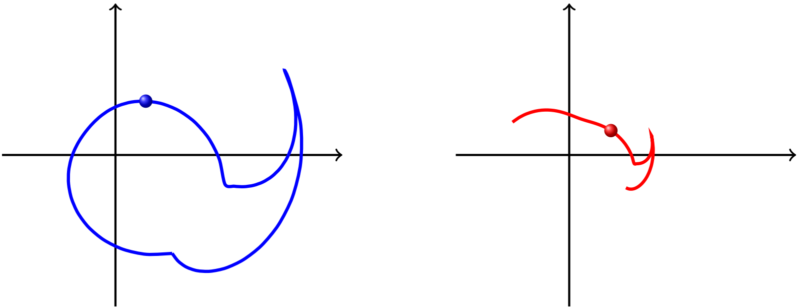

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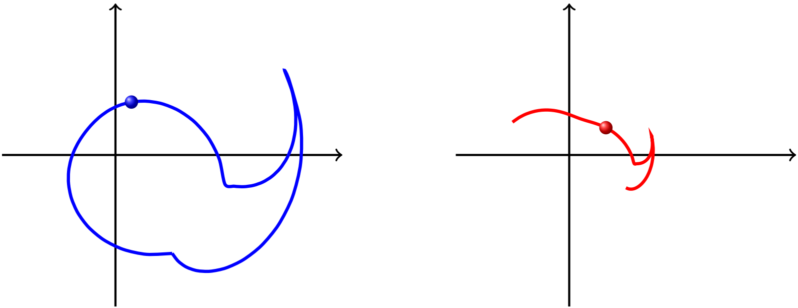

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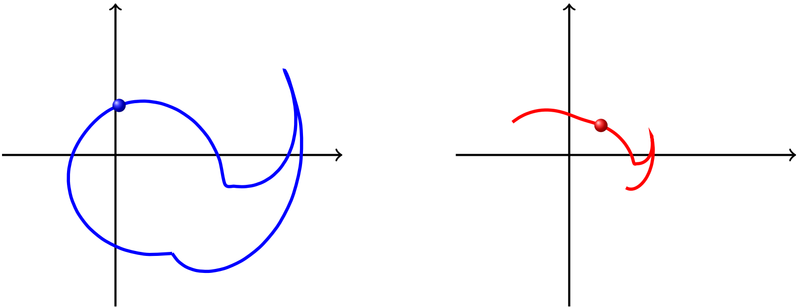

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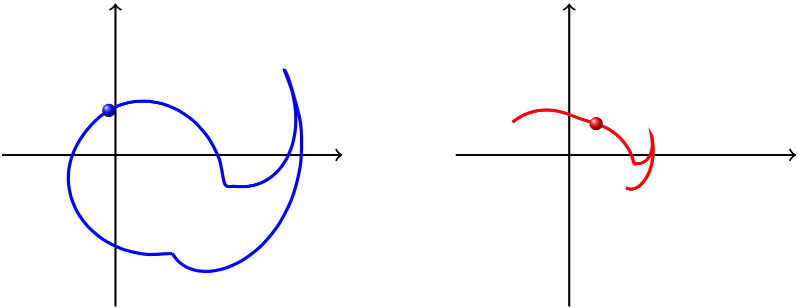

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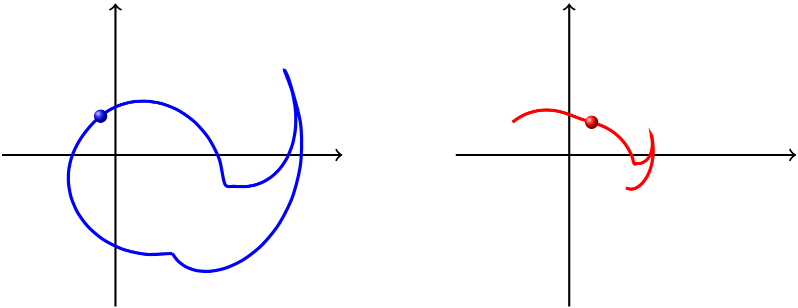

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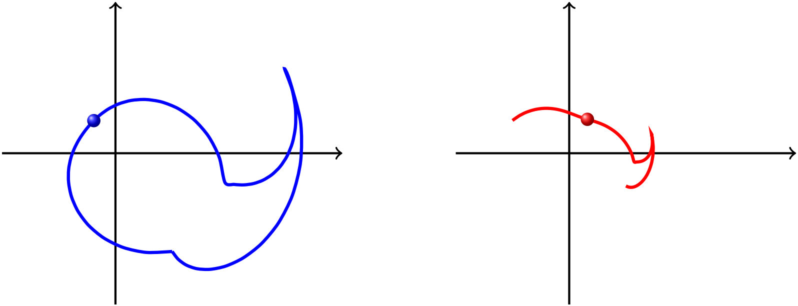

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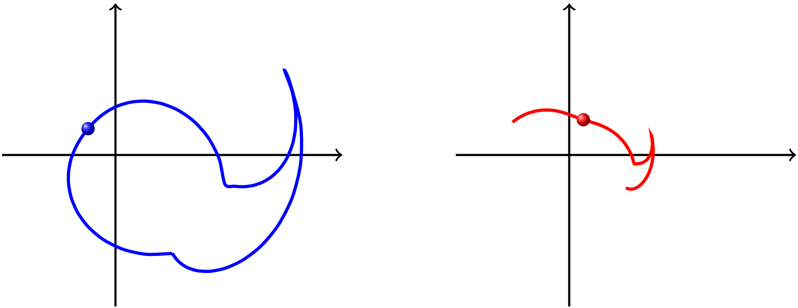

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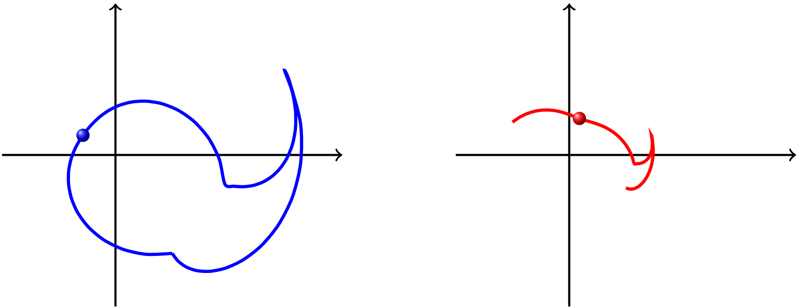

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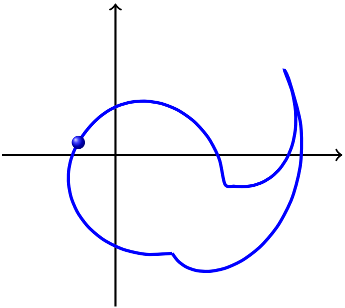

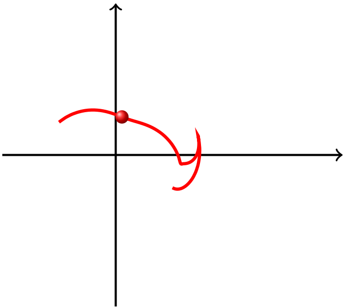

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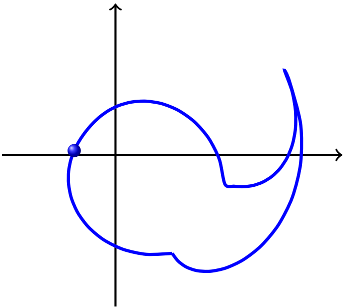

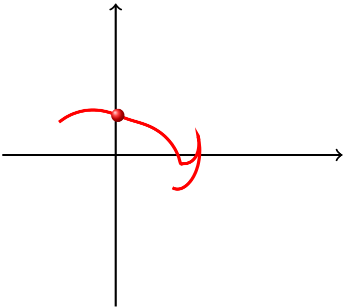

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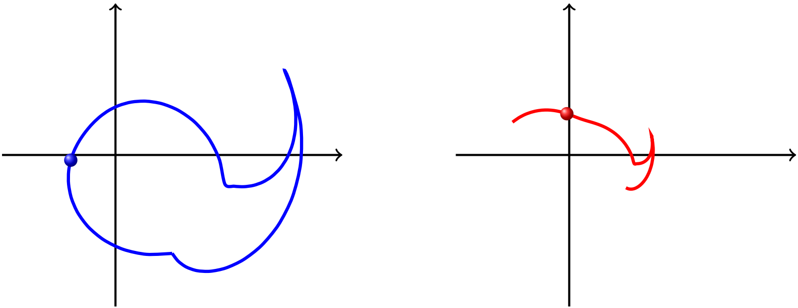

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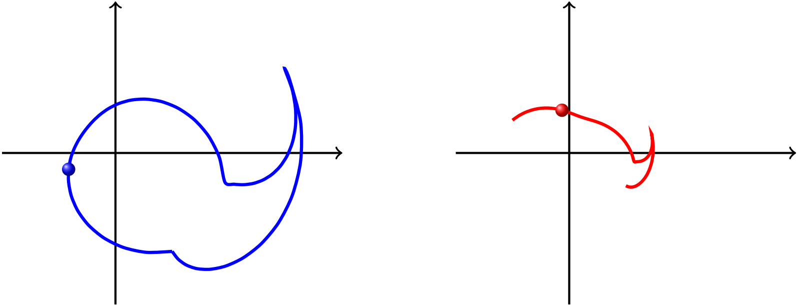

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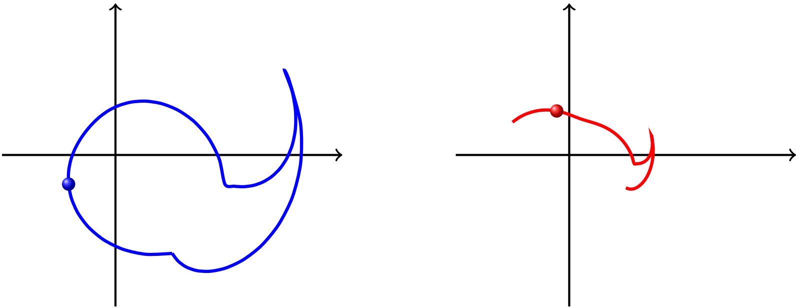

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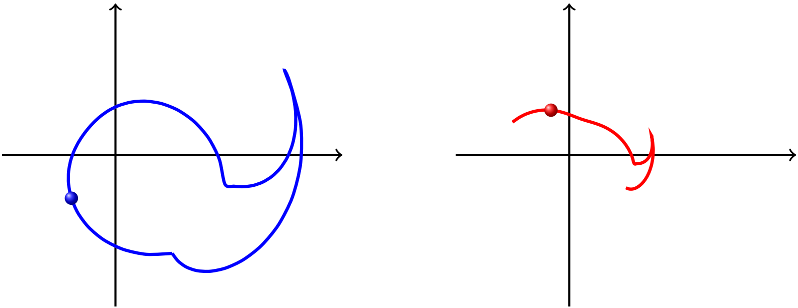

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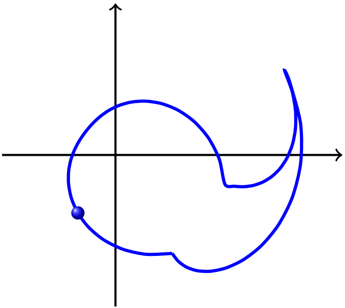

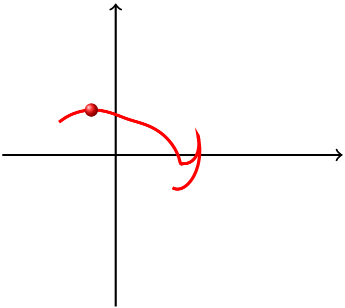

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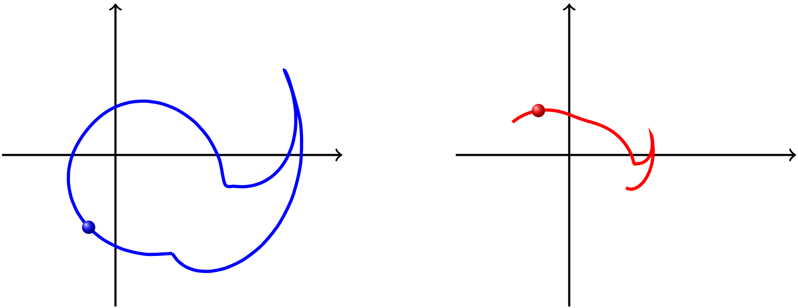

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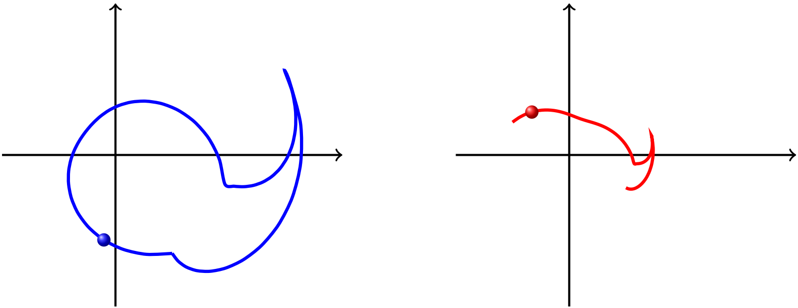

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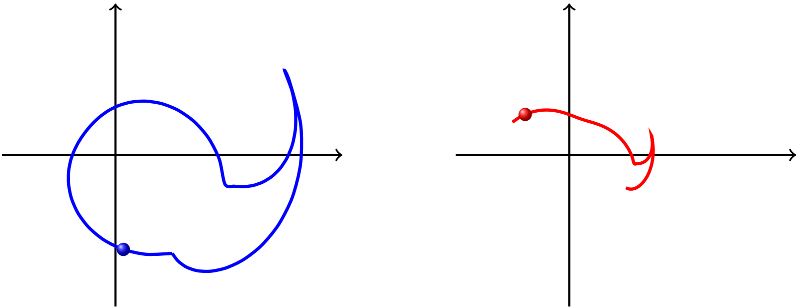

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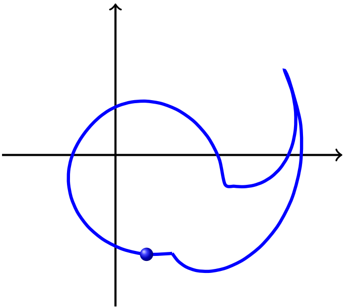

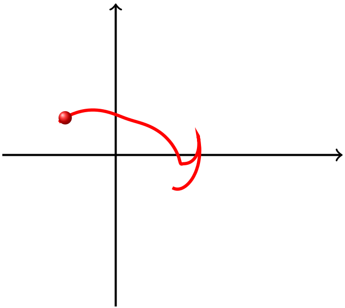

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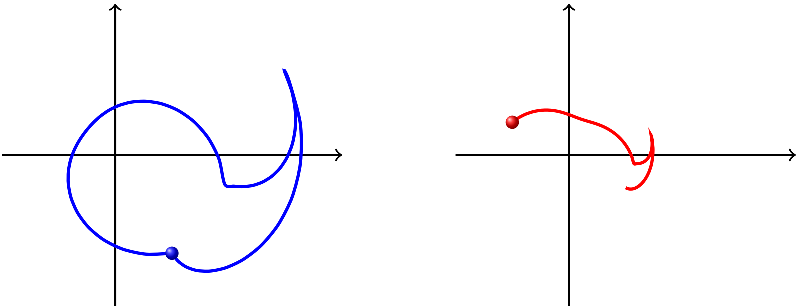

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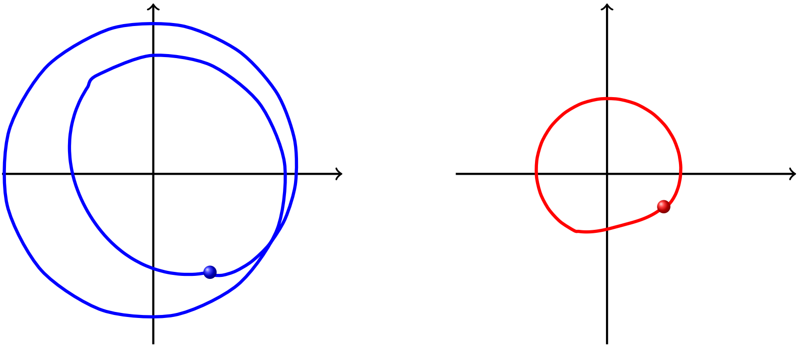

If the closed path encloses the origin, winding round it twice, the value of $\arg(z)$ changes by 4π, and thus that of $\arg\left(z^{\frac{1}{2}}\right)$ changes by 2π.

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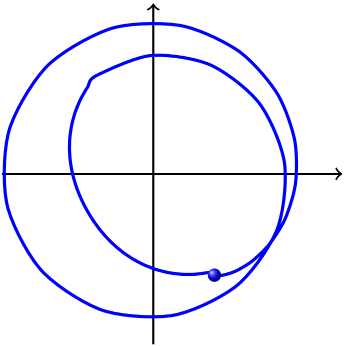

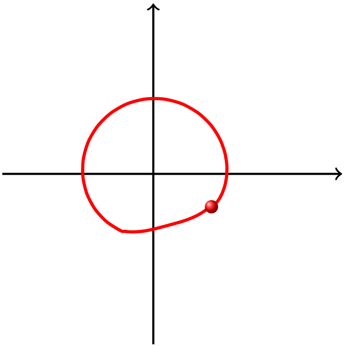

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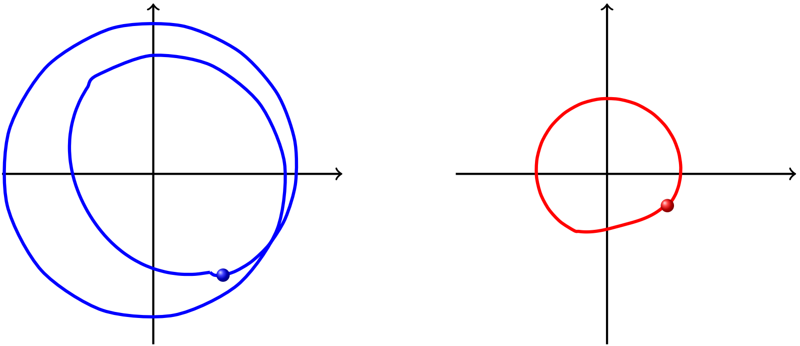

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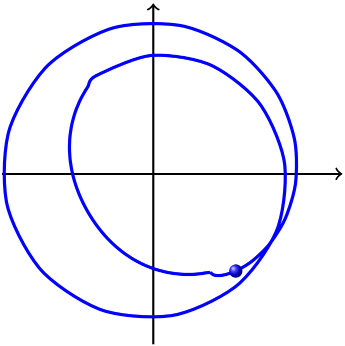

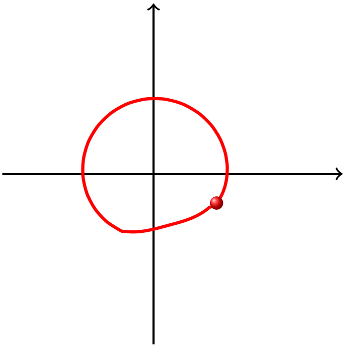

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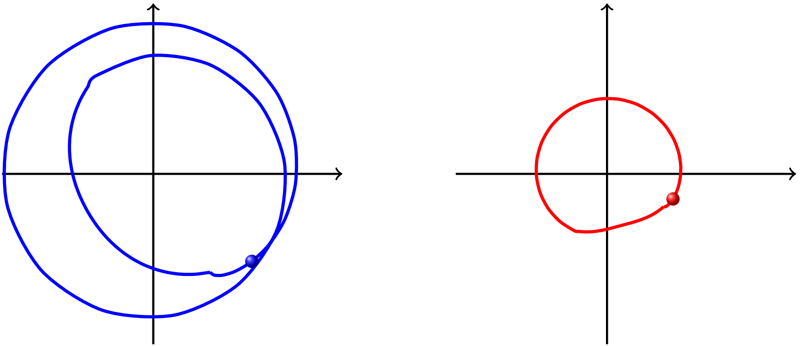

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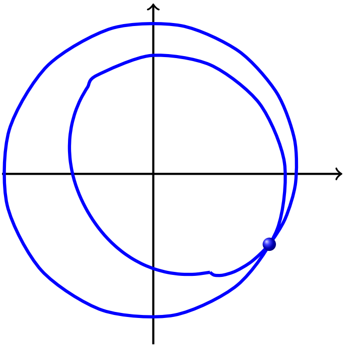

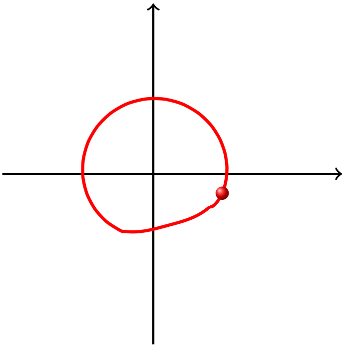

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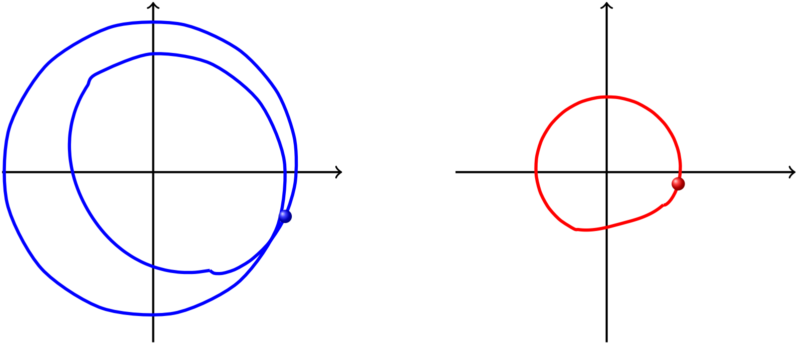

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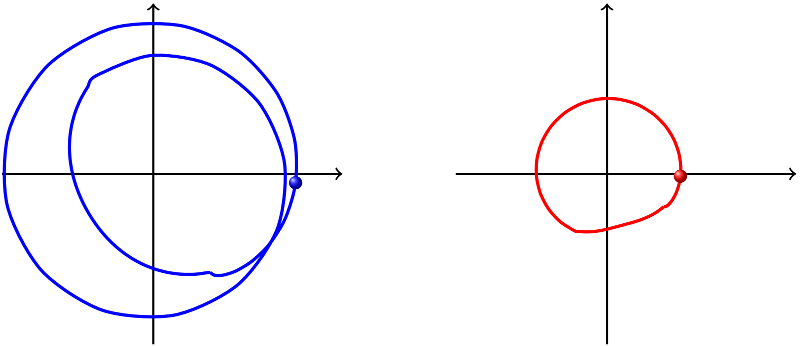

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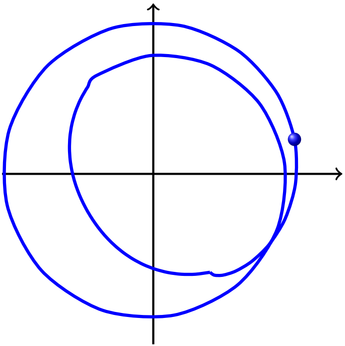

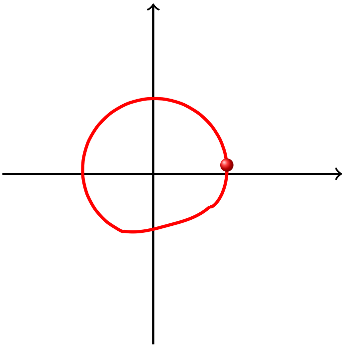

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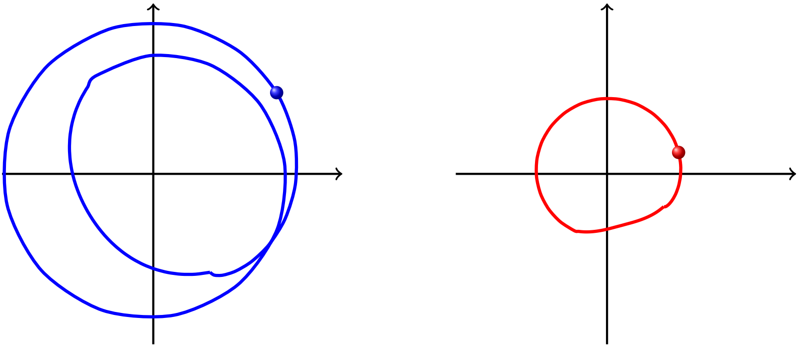

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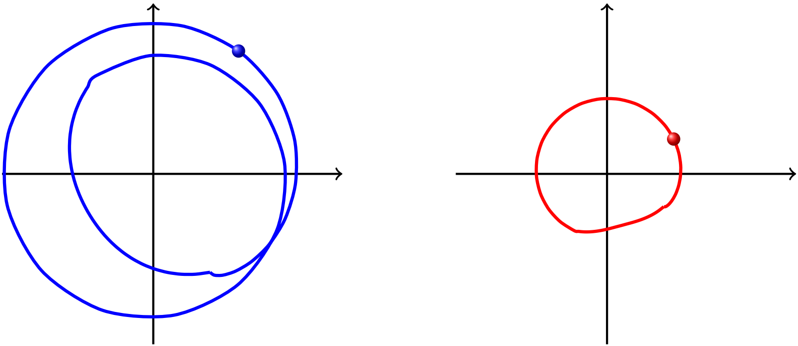

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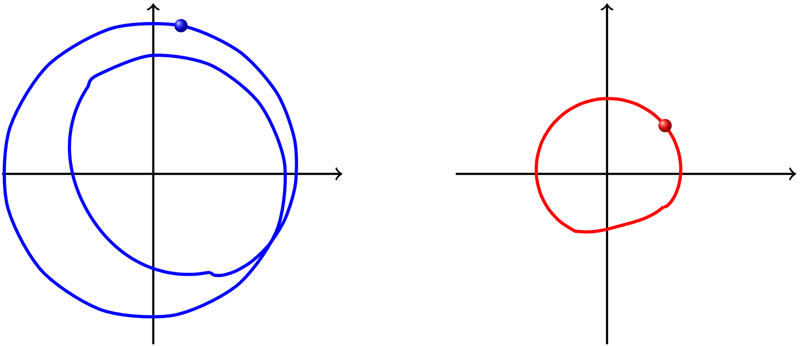

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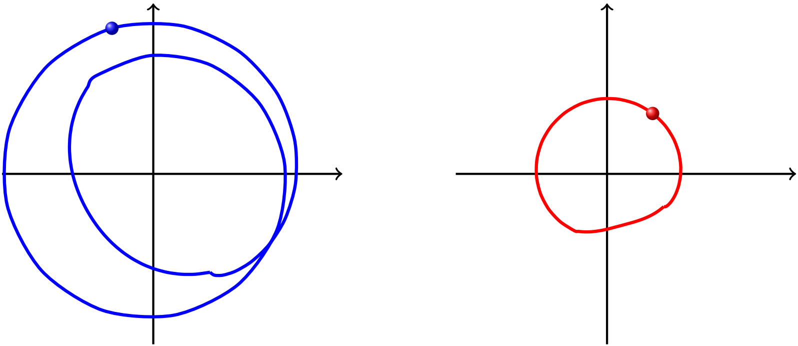

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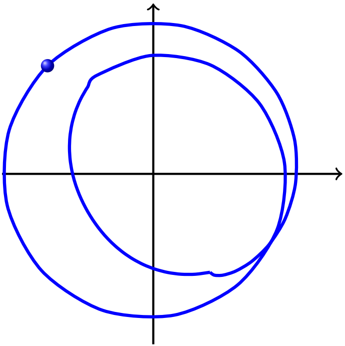

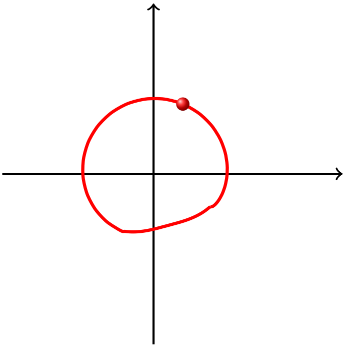

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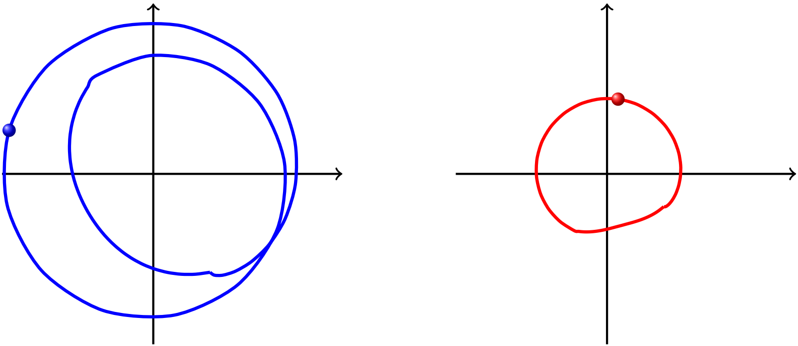

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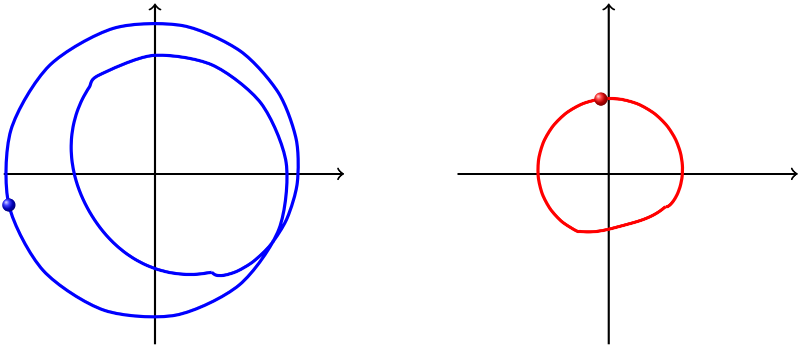

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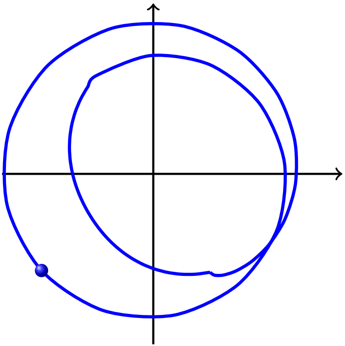

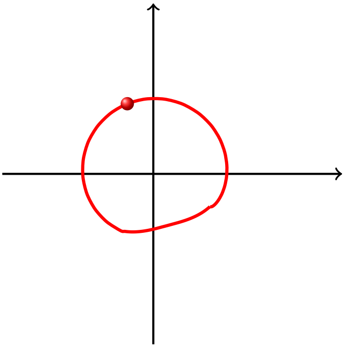

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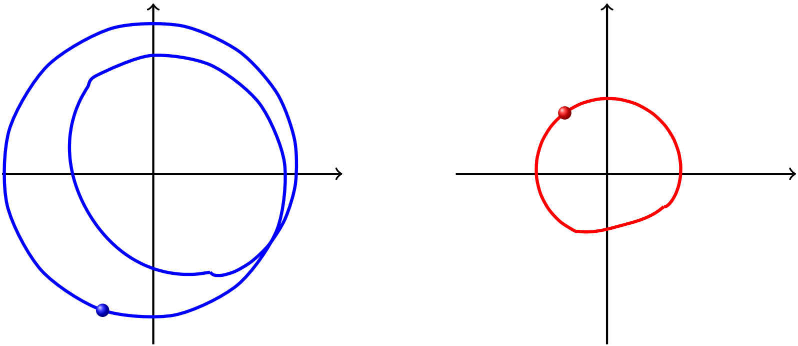

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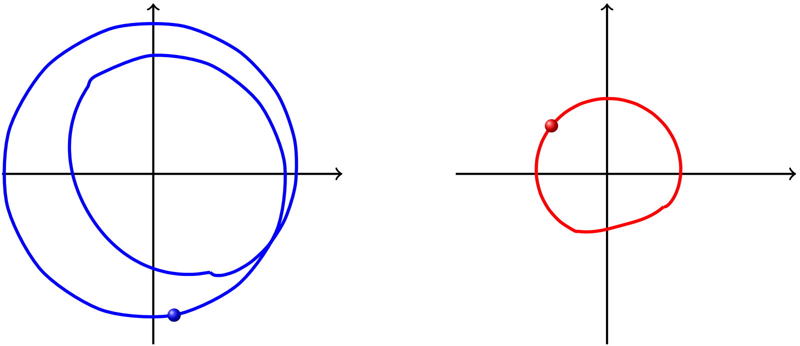

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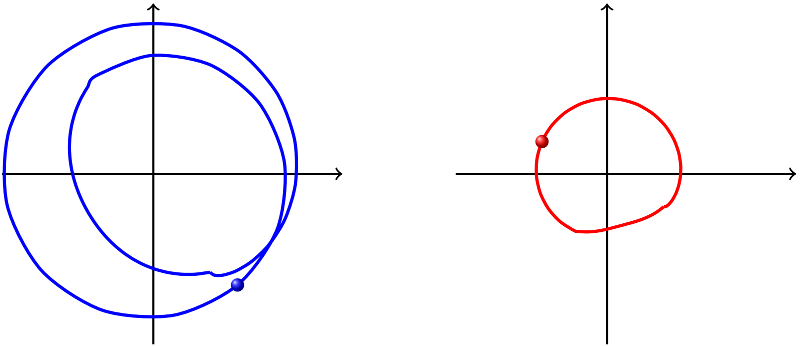

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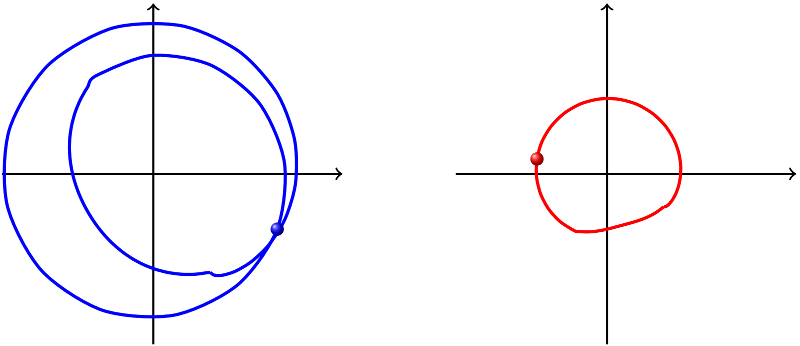

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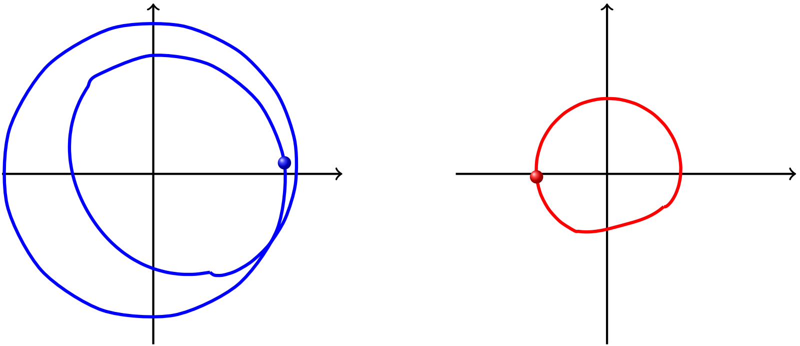

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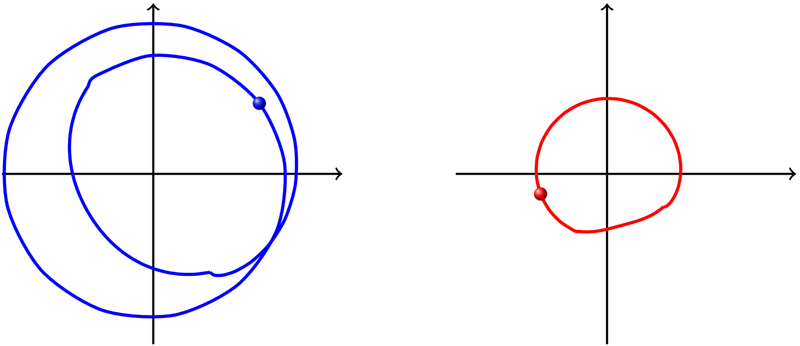

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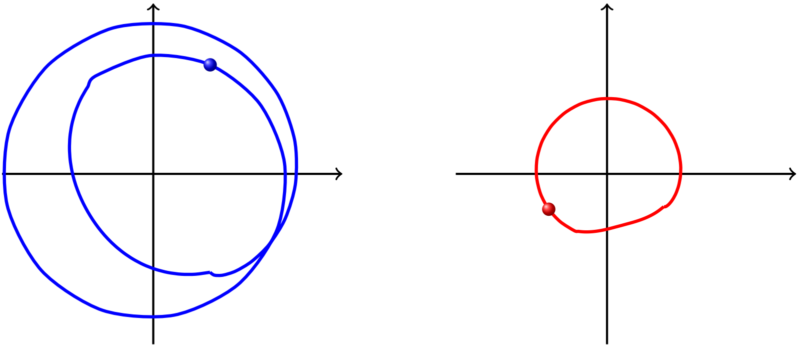

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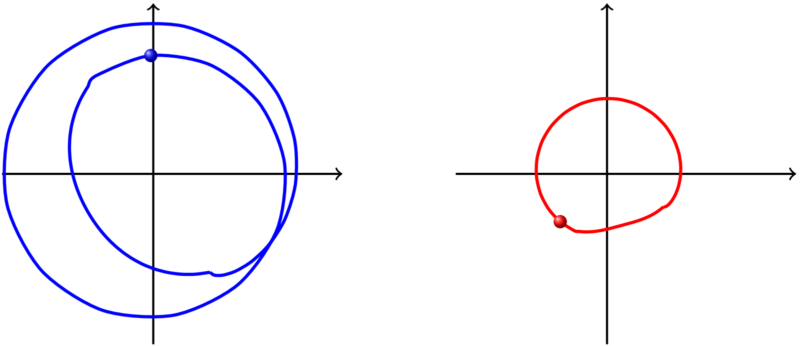

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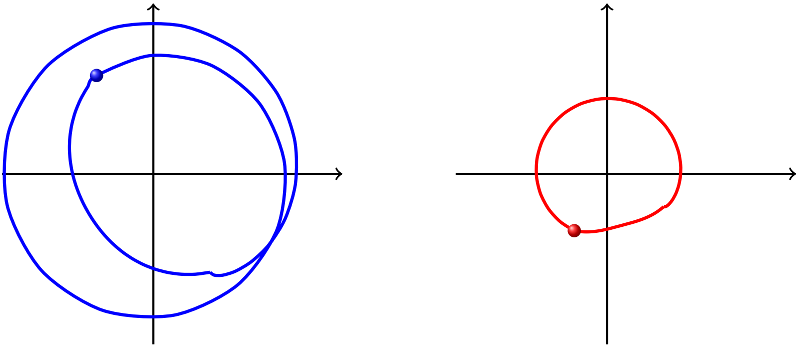

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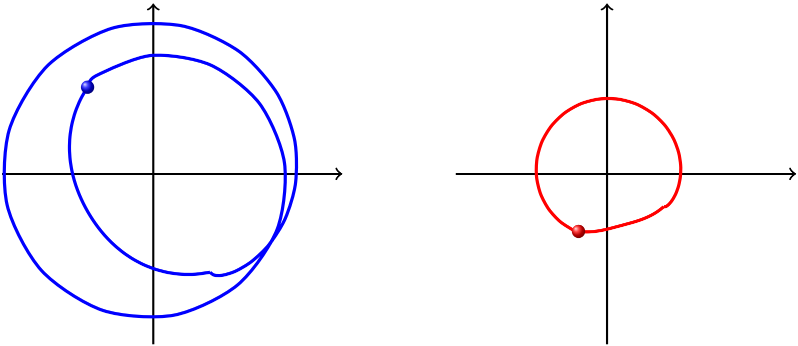

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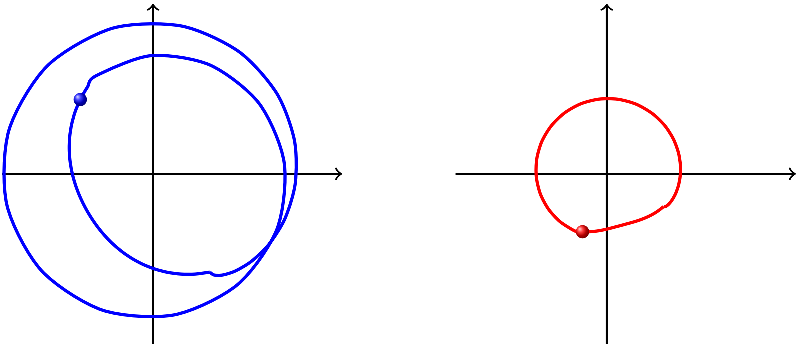

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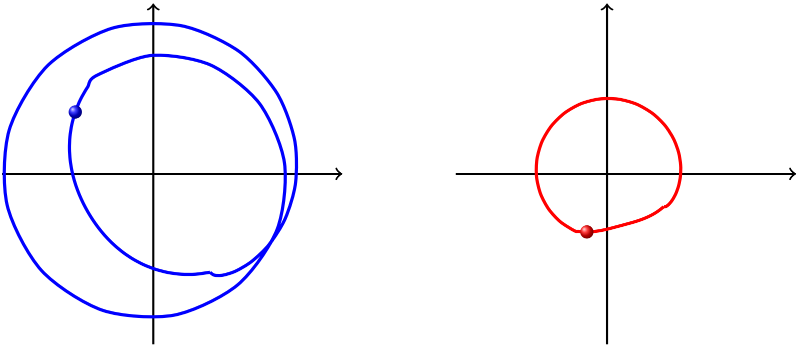

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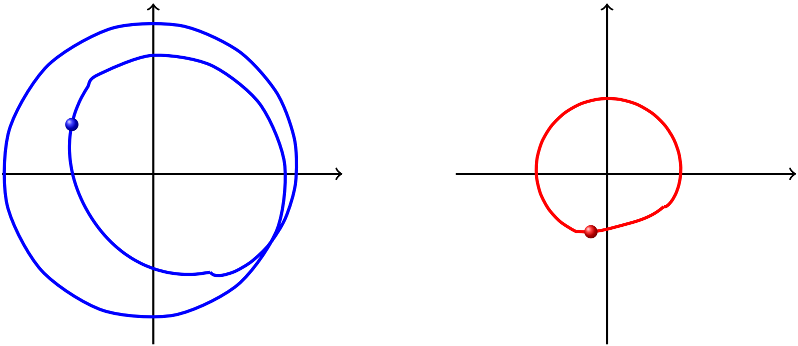

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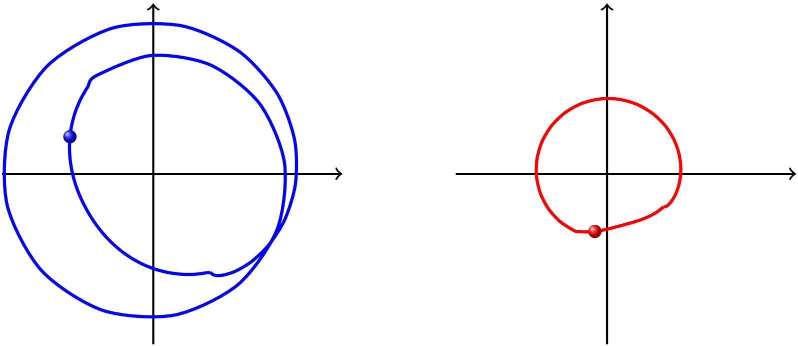

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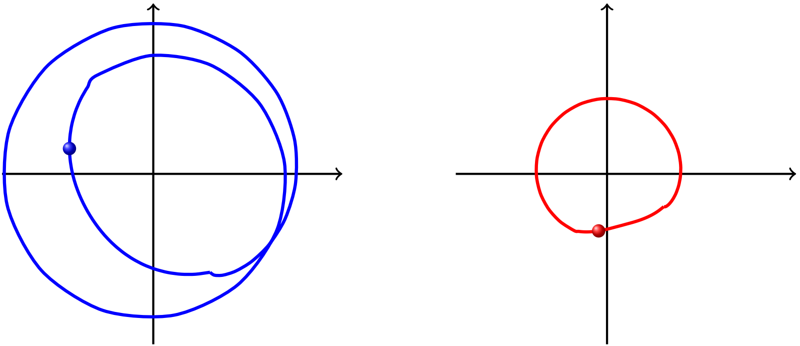

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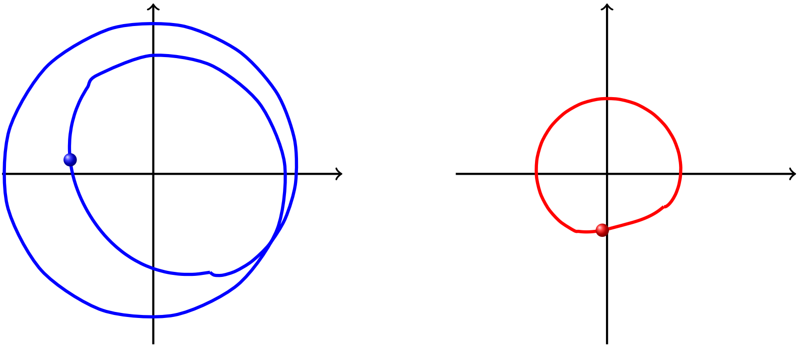

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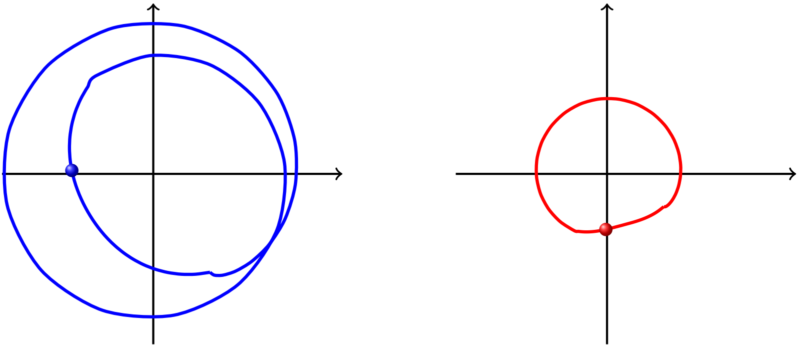

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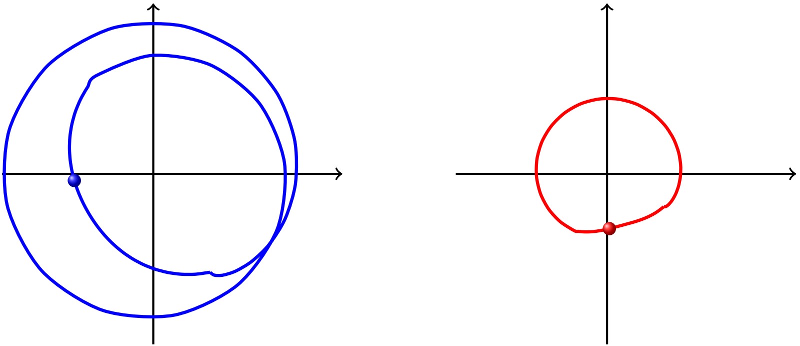

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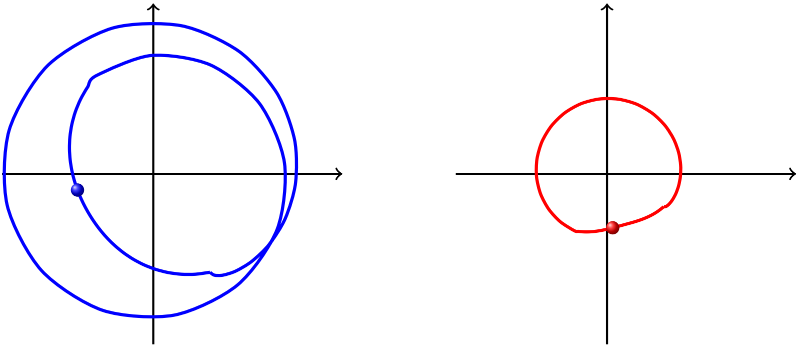

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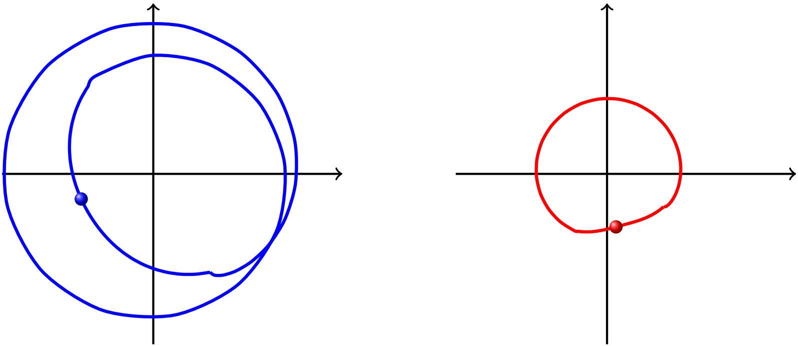

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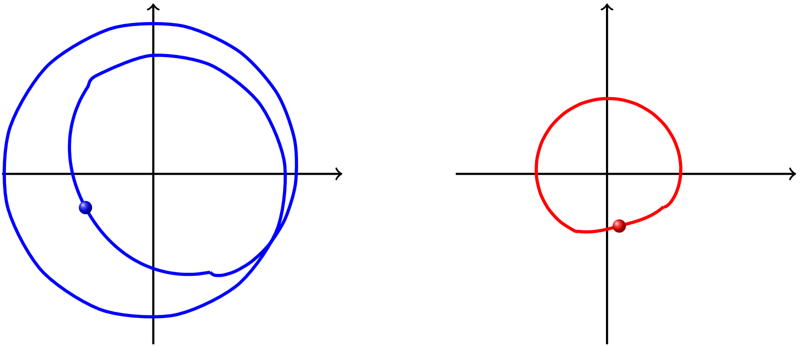

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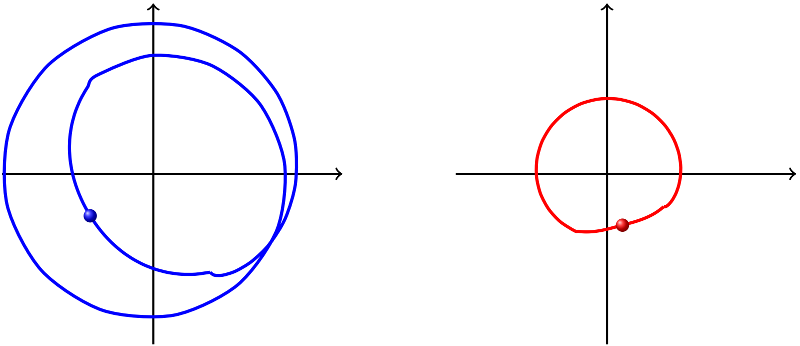

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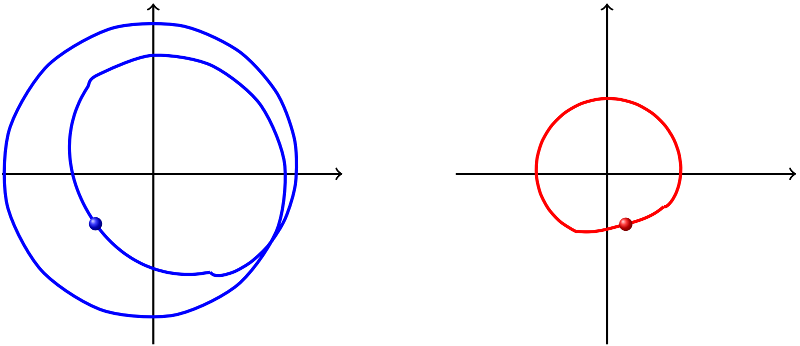

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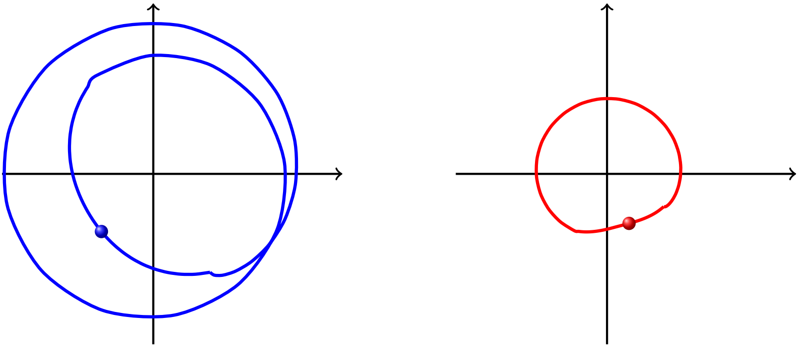

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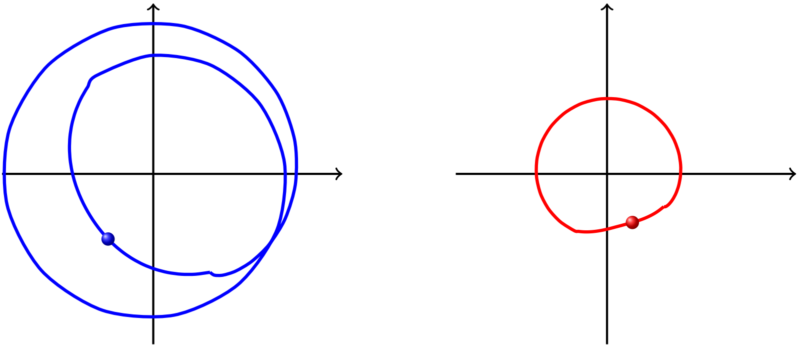

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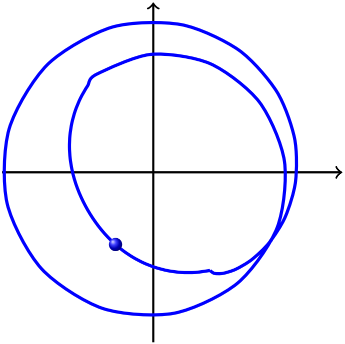

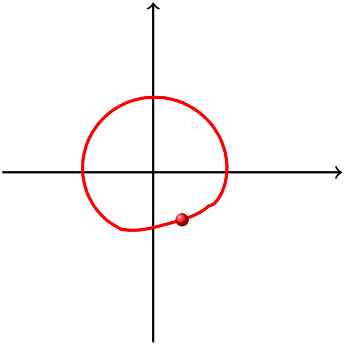

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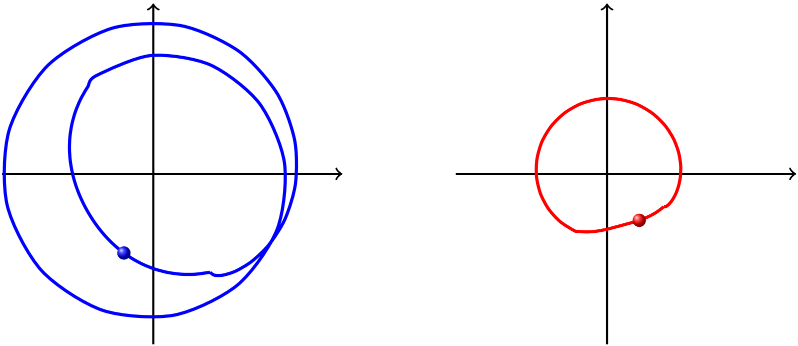

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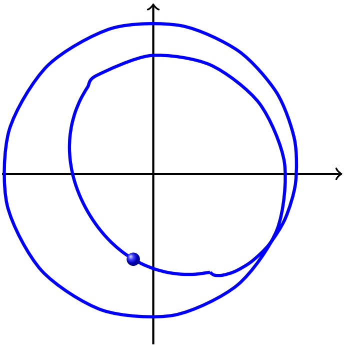

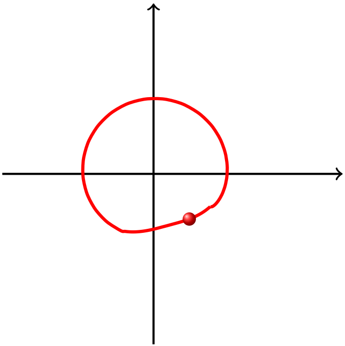

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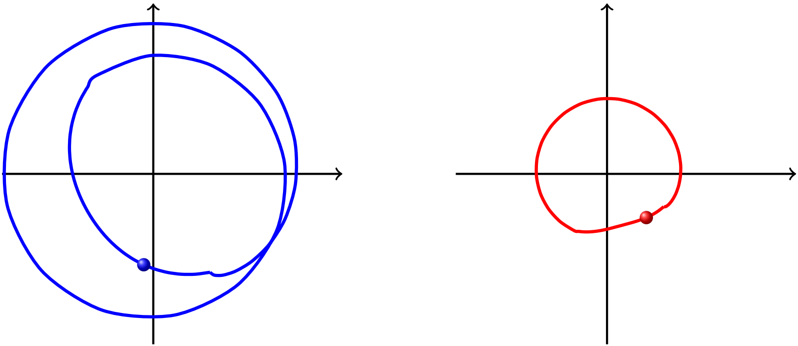

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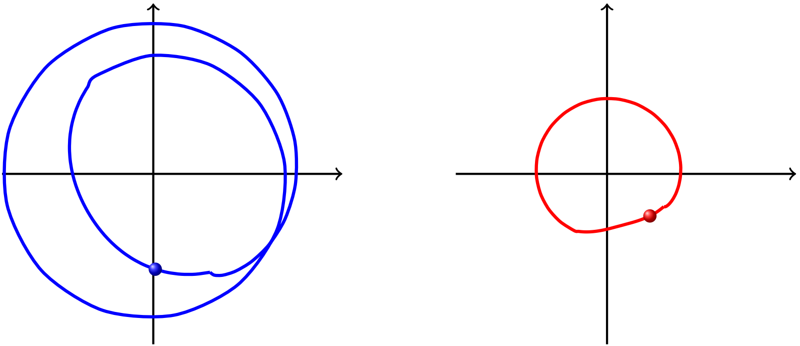

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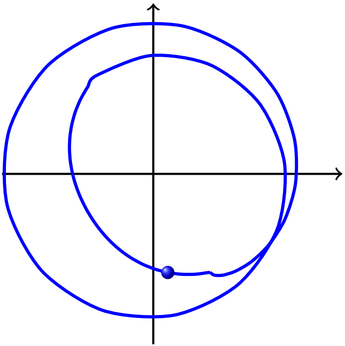

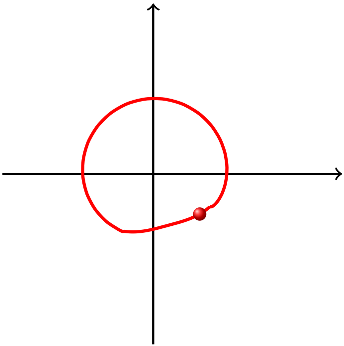

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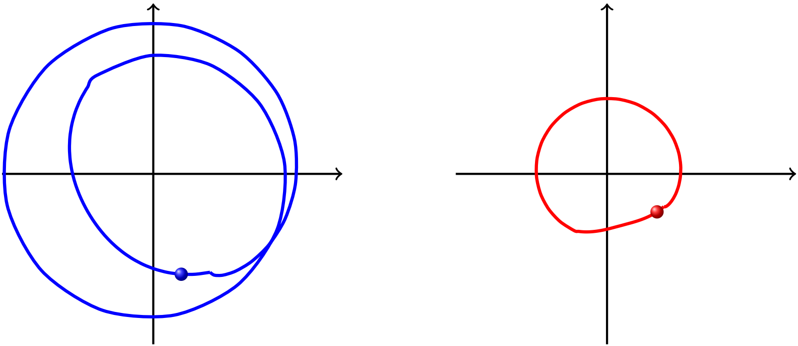

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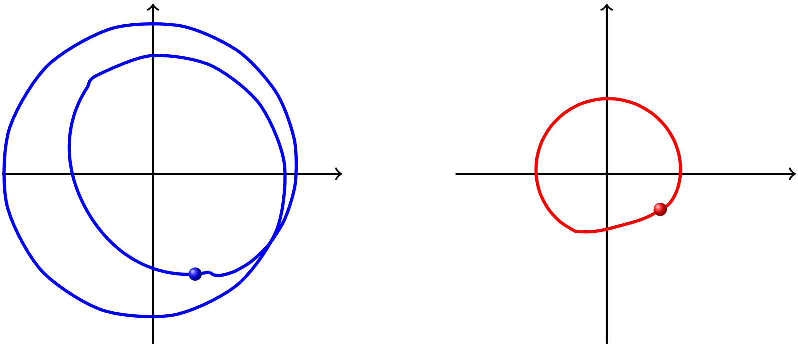

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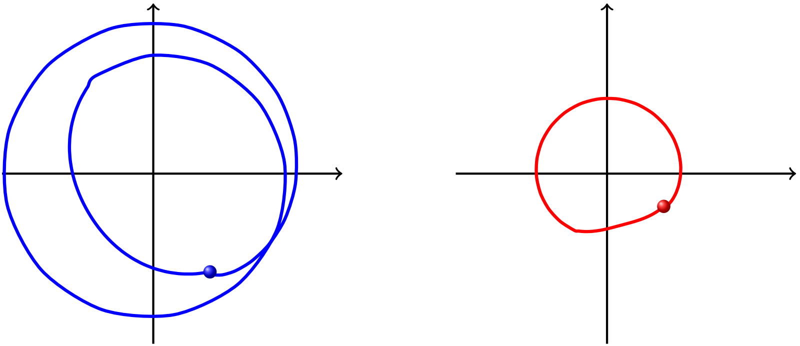

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A multifunction can be then thought of as a set of such branches

$$f(z) = \{f_i(z),\ i = 1, 2, \ldots, n\}$$

where each $z \mapsto f_i(z)$ is a genuine function in the sense of calculus.

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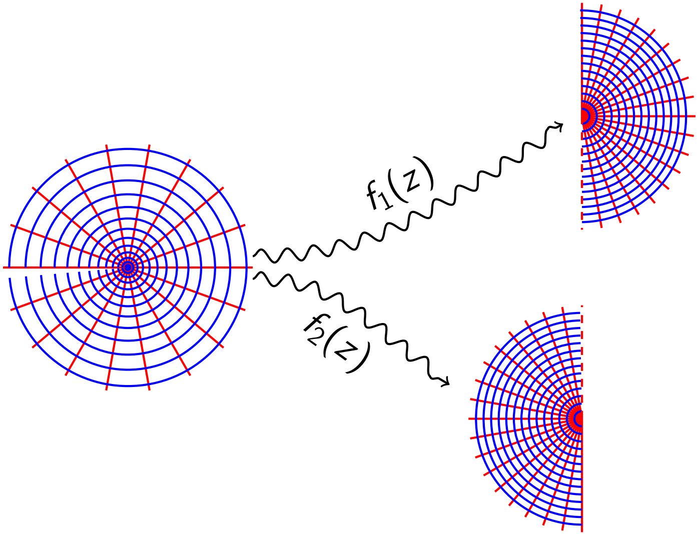

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- Both leads to single valued continuous functions.

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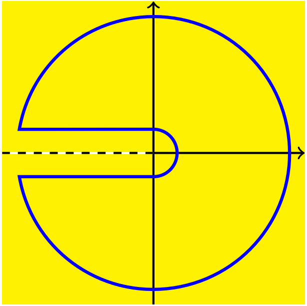

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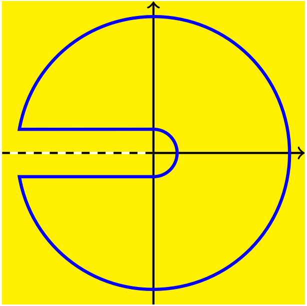

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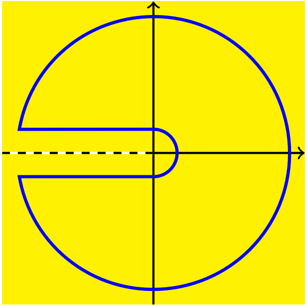

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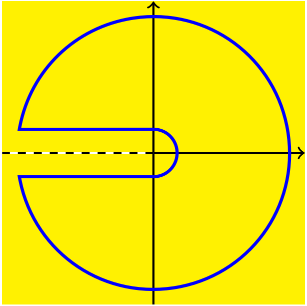

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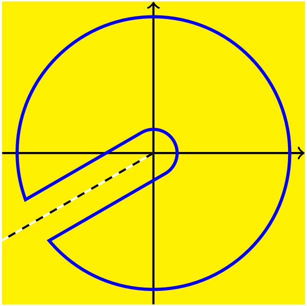
- Each function $f_1, \ f_2$ are discontinuous across the half line $\theta = \alpha$.
- In this case the branch cut is the half line $\theta = \alpha$.
- There are infinite number of ways of defining the branches - one for each value of α.

Dealing with multifunctions



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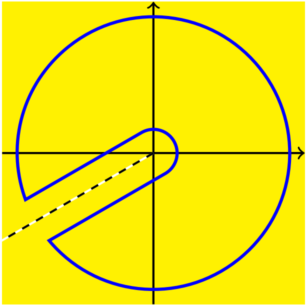

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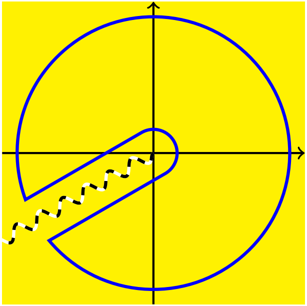

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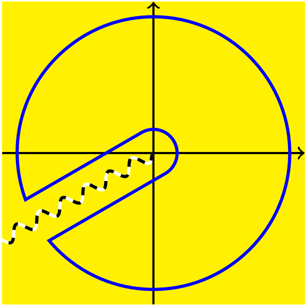

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- Indeed, α need not even be a constant!
- We could even use a different interval $\theta \in (\alpha(r), \alpha(r) + 2\pi]$ for each value of r.
- Thus the branch cut will be the line $\{z = re^{i\theta} \ : \ r > 0, \theta = \alpha(r)\}$

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- It is not possible to encircle the branch point once without either encountering a discontinuity or entering another branch.

Branch points

For a single valued function - traversing a closed contour **once** makes the function return to the original value.

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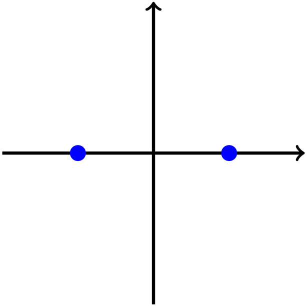
- If the function never returns to the original value, the branch point is called a *logarithmic* branch point.

A case with multiple branch points

$$z \mapsto \sqrt{z^2 - 1} = (z-1)^{\frac{1}{2}}(z+1)^{\frac{1}{2}}$$

- The function has two branch points : $z = 1$ and $z = -1$.



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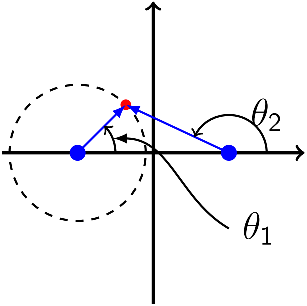

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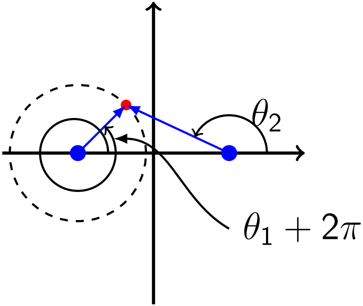

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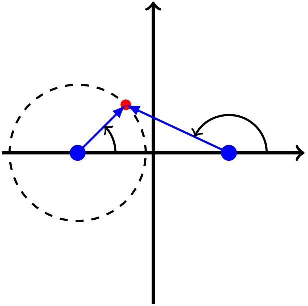

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- $f(z)$ picks up a minus sign.

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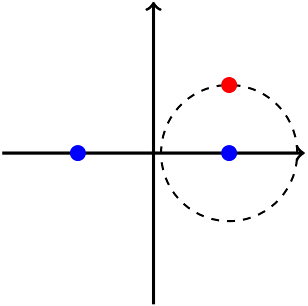

- The function has two branch points : $z = 1$ and $z = -1$.
- Consider a closed curve enclosing $z = -1$.
- After traversing the curve, $\theta_1 = \arg(z+1)$ changes by 2π, while $\theta_2 = \arg(z-1)$ does not change.
- $f(z)$ picks up a minus sign.
- The situation is similar for a curve enclosing $z = +1$.

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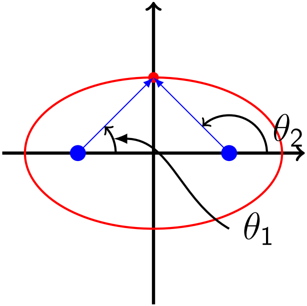

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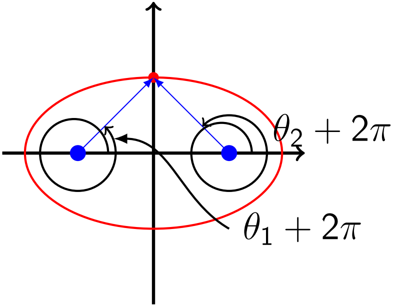

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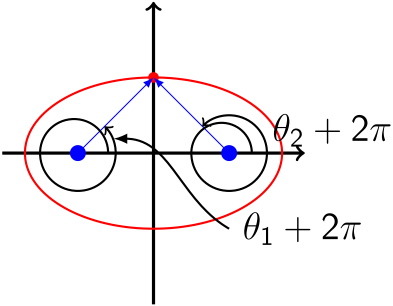

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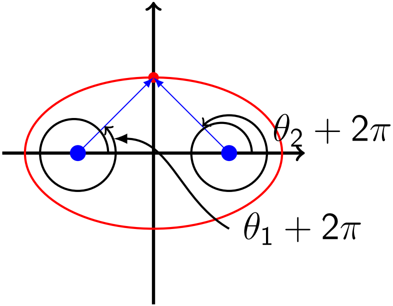

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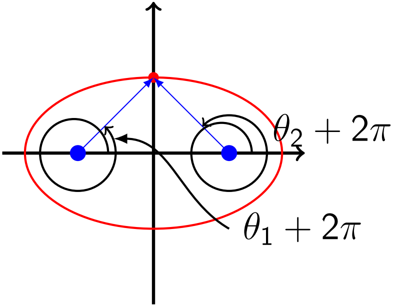

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- $\arg\left(z^2 - 1\right)$ increases by 4π.
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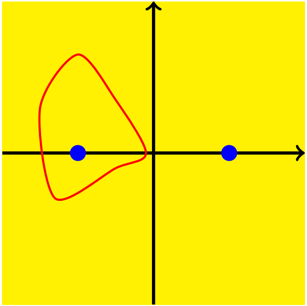
- The multifunction stays at the same value.

Branch cut(s) for $z \mapsto \sqrt{z^2 - 1}$



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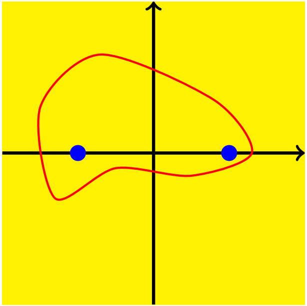

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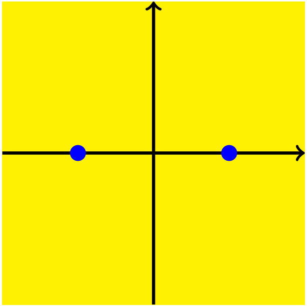

The multifunction enters another branch whenever a point is transported along a closed curve that goes around *either* of the two branch points $z = \pm 1$ once - but not when the closed curve goes around *both* once! The job of branch cuts is to prevent it from entering this new branch.

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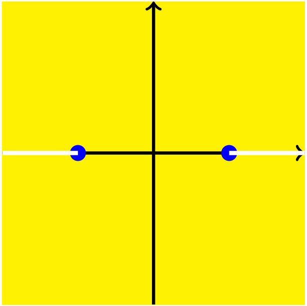

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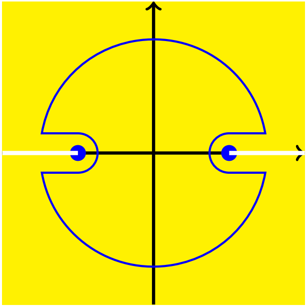

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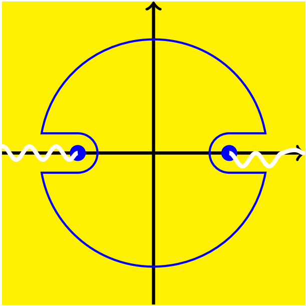

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The branch cut does not *have* to be straight!

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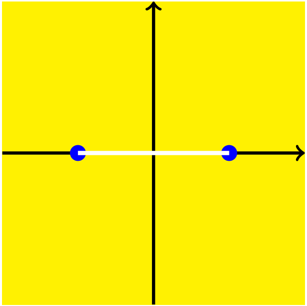

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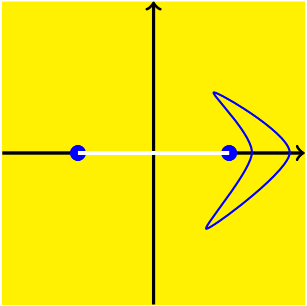

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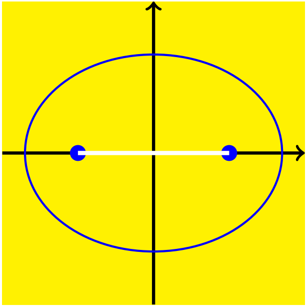

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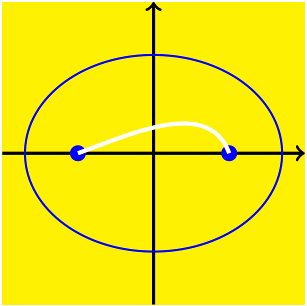

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Any curve joining $z = -1$ with $z = +1$ will do!

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A Mobius transformation is a mapping of the form

$$z \mapsto M(z) = \frac{az + b}{cz + d}$$

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- A Mobius transformation for which $ad - bc \neq 0$ is called *non-singular*.

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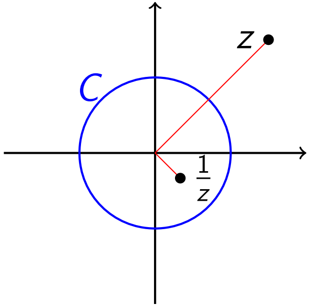
Complex Inversion



The image of $z = re^{i\theta}$ under complex inversion is

$$z \mapsto \frac{1}{z} = \frac{1}{r}e^{-i\theta}$$

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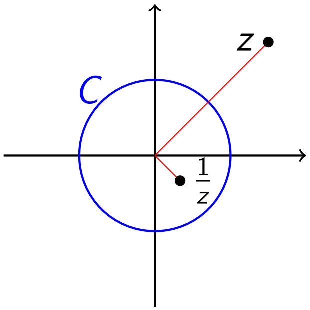

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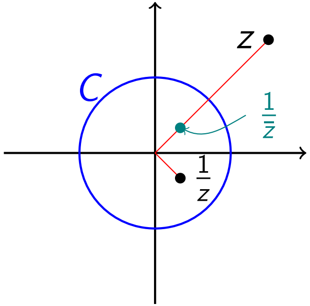

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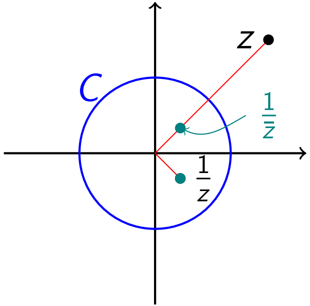

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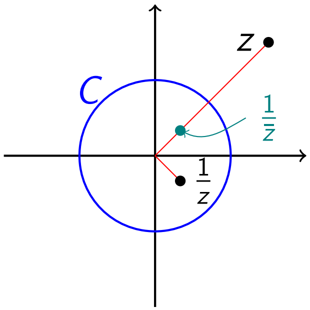

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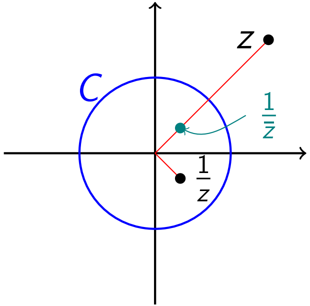

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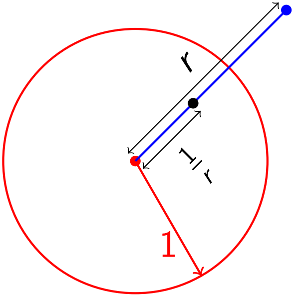
Complex conjugation has a trivial geometry - so all the excitement comes from the geometric inversion!

Geometric Inversion



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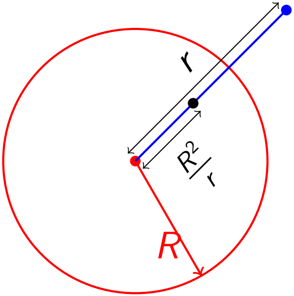
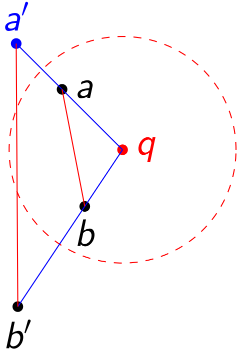


- The map $z \mapsto \frac{1}{\bar{z}}$ is the same as $\mathcal{I}_C$ - inverion in a unit circle.
- General inversion $\mathcal{I}_K$ in a circle K of radius R maps a point P at distance r from the origin Q to another at a distance $\frac{R^2}{r}$, on the line joining P to O.

Geometry of inversions:

Theorem

If inversion in a circle centered at q maps two points a and b into a' and b', respectively, then the triangles aqb and $b'qa'$ are similar.



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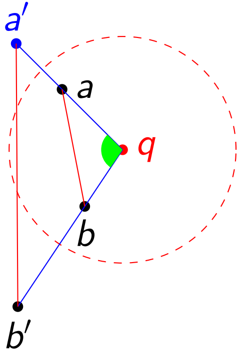

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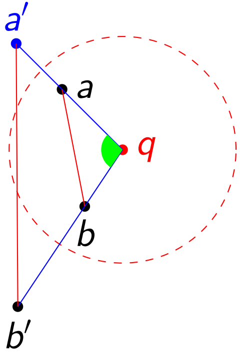

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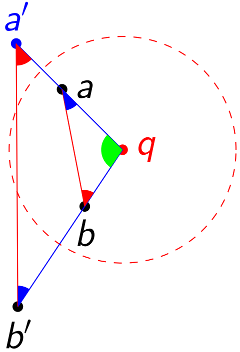

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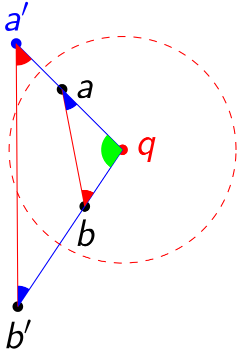

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- The two triangles are similar!

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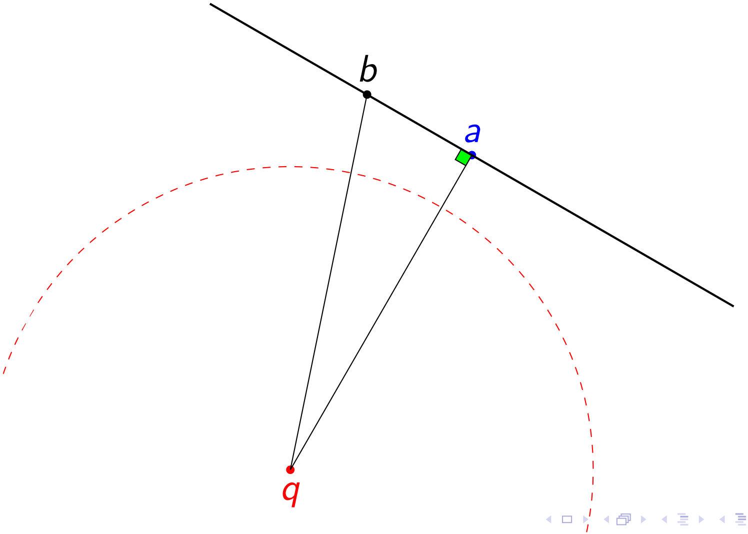
- This is a hint of the deep connection between the Riemann sphere and Mobius transformations!

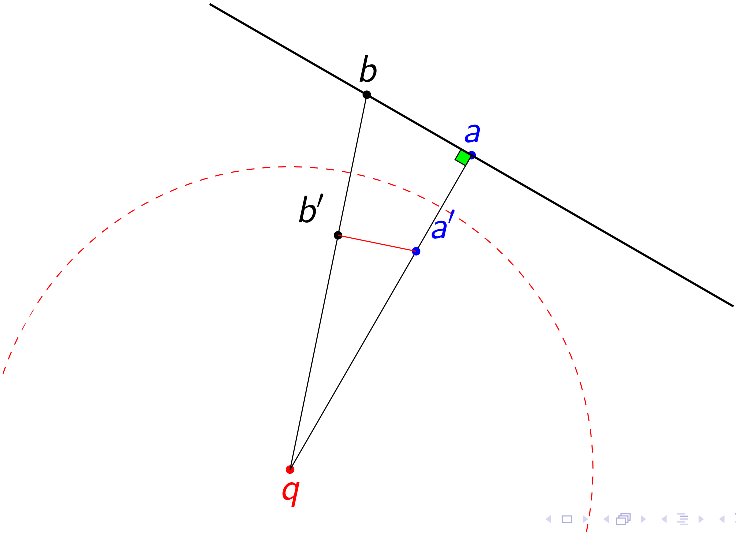
Straight line $\mapsto$ a circle!

b is an arbitary point on a straight line whose point closest to q is a.



Straight line $\mapsto$ a circle!

The triangles abq and $b'qa'$ are similar.



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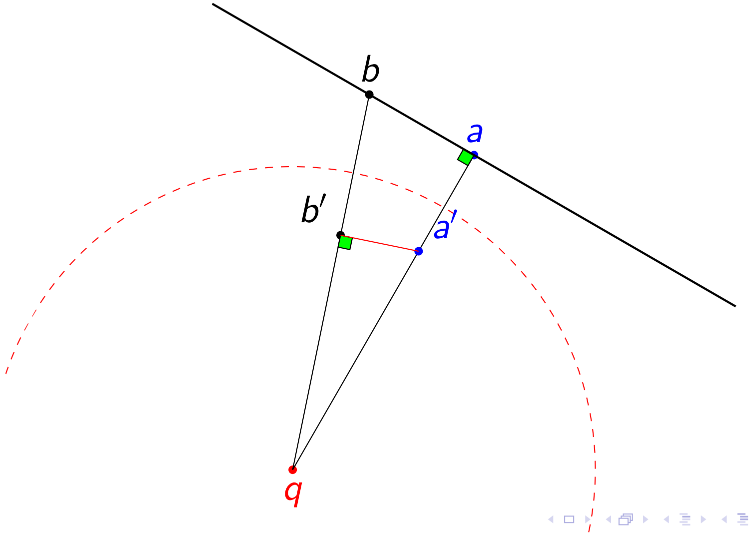
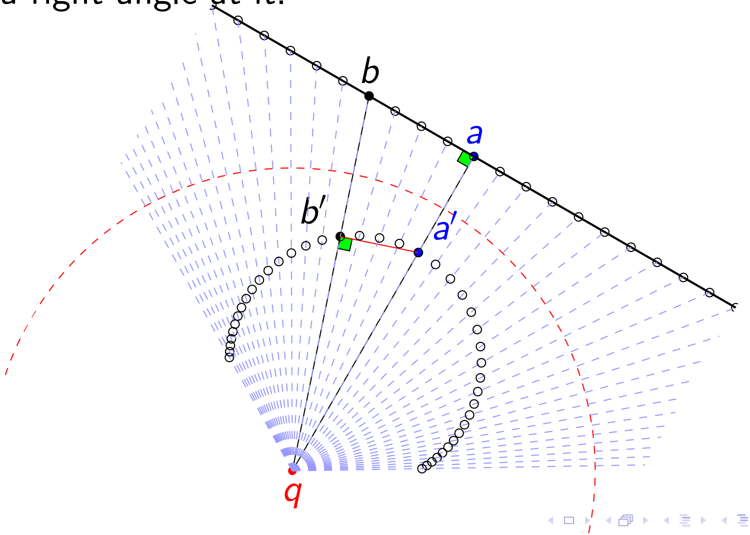
The angle $qb'a'$ is equal to qab - a right angle.



Straight line ↦ a circle!

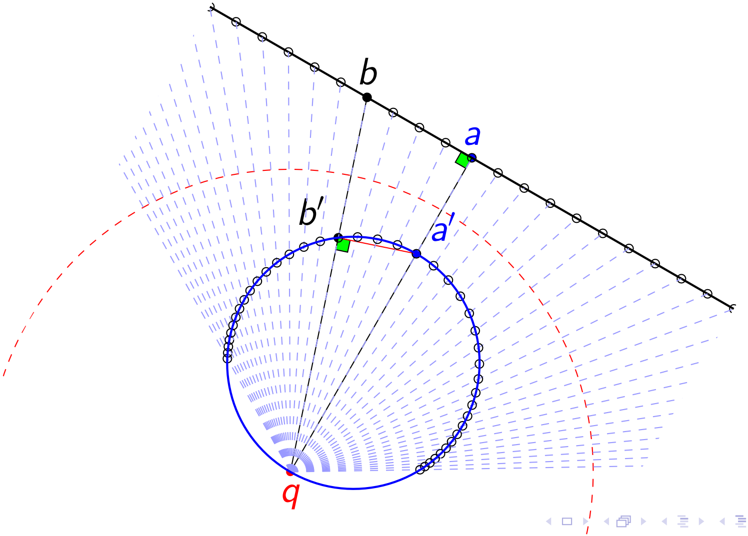
As the point b moves along the straight line, the point b' always moves in a path such that qa' always subtend a right angle at it.



Straight line $\mapsto$ a circle!

The locus of b' is a circle with qa' as diameter!

◂ Go Back!



Why restrict θ to $(-\pi, \pi]$?

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Go Back!

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Go Back!

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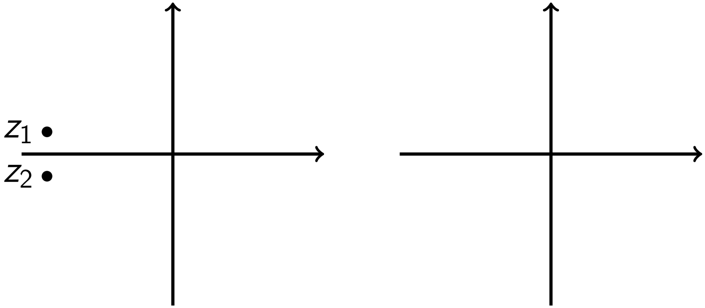
The ambiguity can be lifted by keeping θ restricted to any semi-closed 2π interval!

Go Back!

Why are f_1 and f_2 discontinuous?

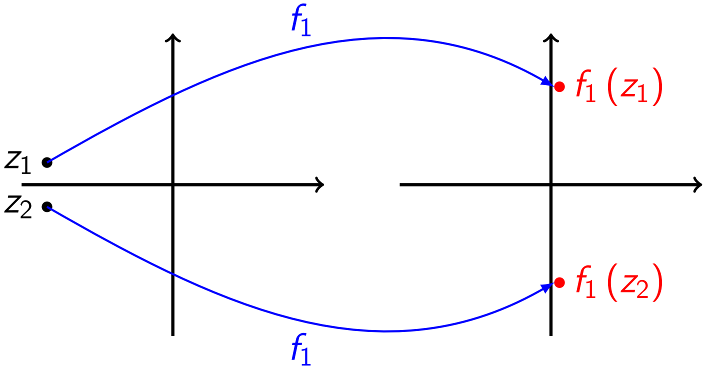
Consider two points $z_1 = re^{\mathrm{i}(\pi-\epsilon)}$ and $z_2 = re^{-\mathrm{i}(\pi-\epsilon)}$ which are very close for small ϵ.



Go Back!

Why are f_1 and f_2 discontinuous?

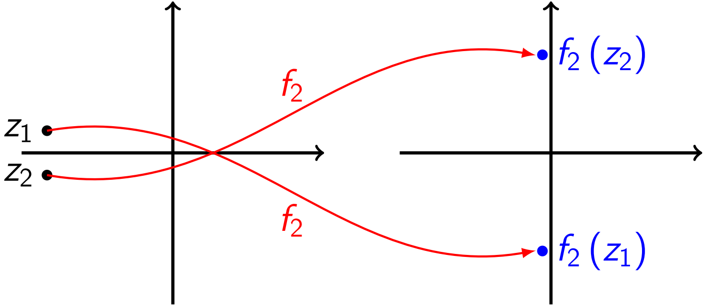
Consider two points $z_1 = re^{\mathrm{i}(\pi-\epsilon)}$ and $z_2 = re^{-\mathrm{i}(\pi-\epsilon)}$ which are very close for small ϵ.



Under f_1 the neraby points z_1 and z_2 map onto far-off points!

Why are f_1 and f_2 discontinuous?

Consider two points $z_1 = re^{\mathrm{i}(\pi-\epsilon)}$ and $z_2 = re^{-\mathrm{i}(\pi-\epsilon)}$ which are very close for small ϵ.



The same thing happens under f_2!

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