Complex Functions : Limits and continuity

Ananda Dasgupta

MA211, Lecture 6

Continuous functions on $\mathbb{R}$

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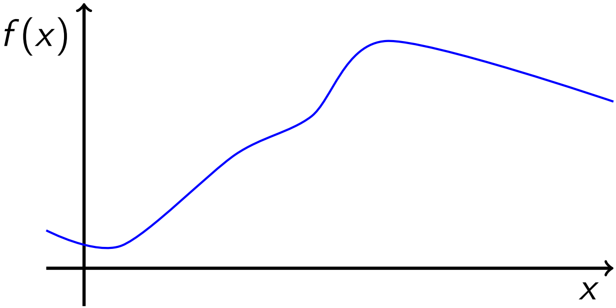
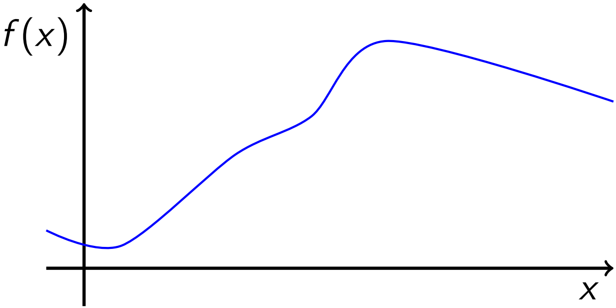


Continuous functions on $\mathbb{R}$

The intuitive notion :

A function whose graph can be drawn without lifting pen from paper.

- Easy to understand.
- Hard to use in rigorous discussions.

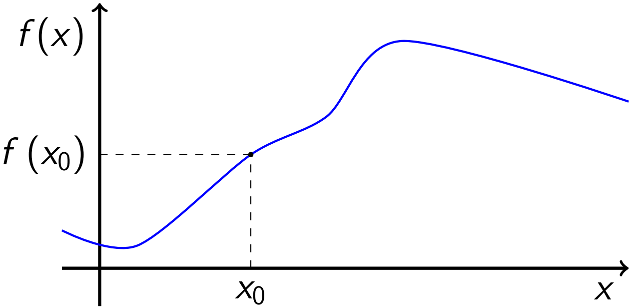


Continuous functions on $\mathbb{R}$

Precise definition :

A function $f : S \subset \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in S$ if $\forall \epsilon > 0$, $\exists \delta > 0$:

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$$



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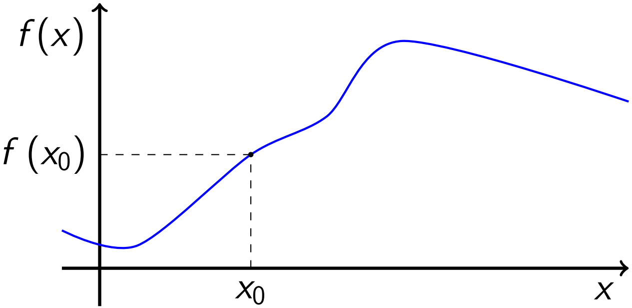
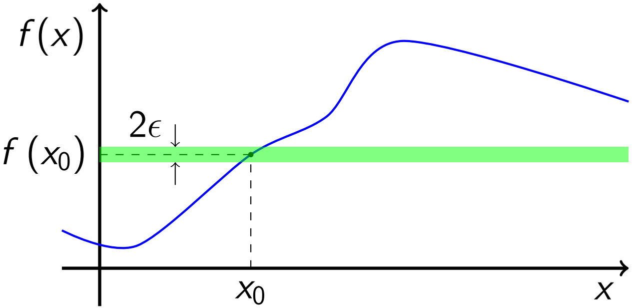
We can prove rigorous results with this.



Continuous functions on $\mathbb{R}$

Geometric meaning

No matter how finicky we are, we can confine the value of $f(x)$ to a narrow enough band centered at $f(x_0)$,



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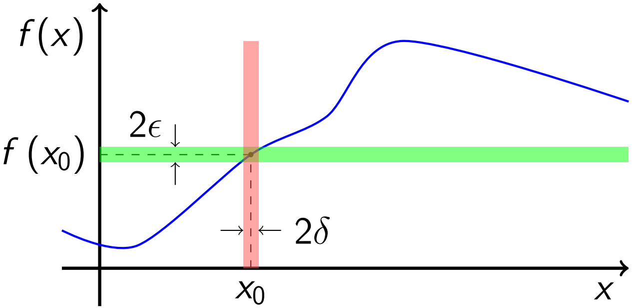
No matter how finicky we are, we can confine the value of $f(x)$ to a narrow enough band centered at $f(x_0)$, by keeping x confined to a sufficiently narrow band centered at x_0.



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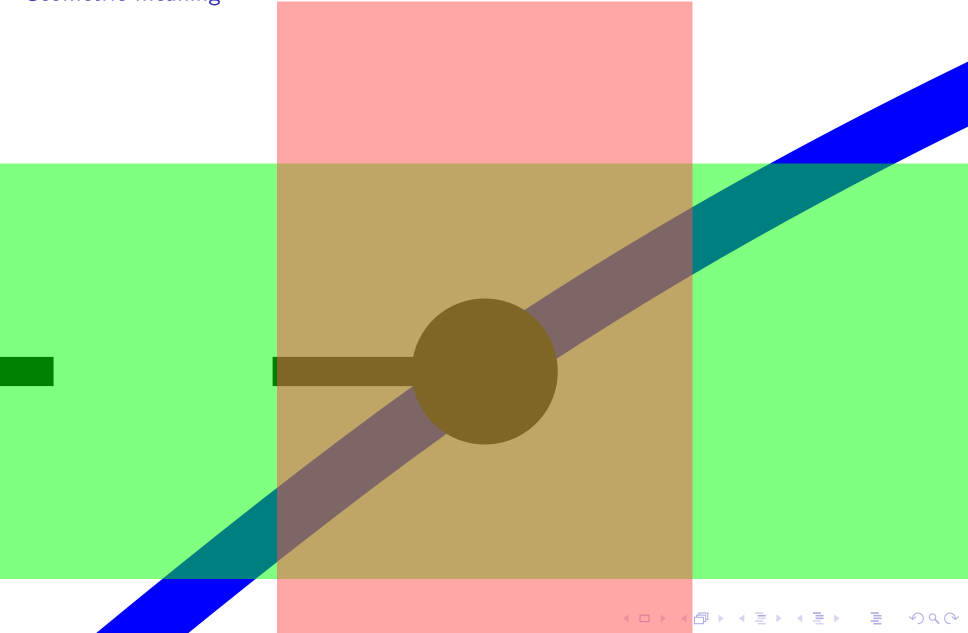
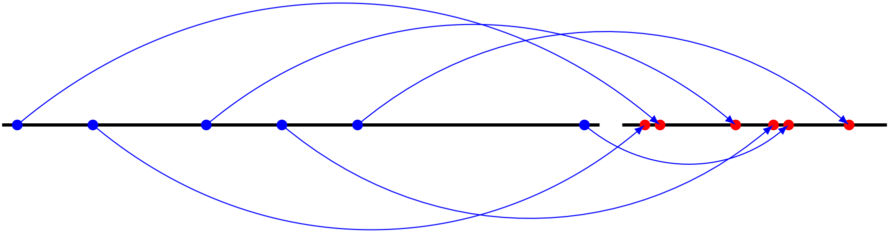


Continuous functions on $\mathbb{R}$

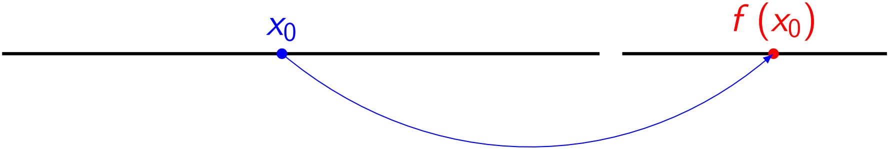
Geometric meaning - a different look



We can picture map $f : x \mapsto f(x)$ as carrying points in one copy of the real line to another.

Continuous functions on $\mathbb{R}$

Geometric meaning - a different look



It carries the point at x_0 to the point at $f(x_0)$.

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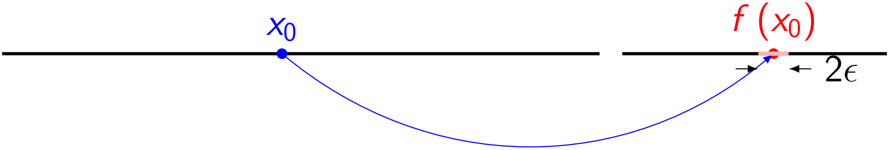

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Consider the interval $(f(x_0) - \epsilon, f(x_0) + \epsilon)$ of width 2ϵ centered around $f(x_0)$.

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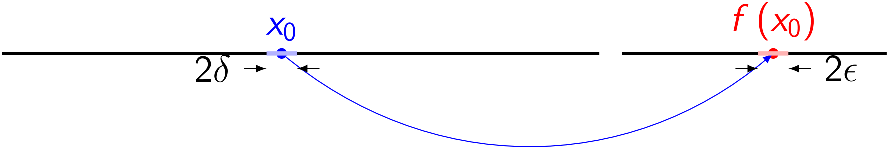

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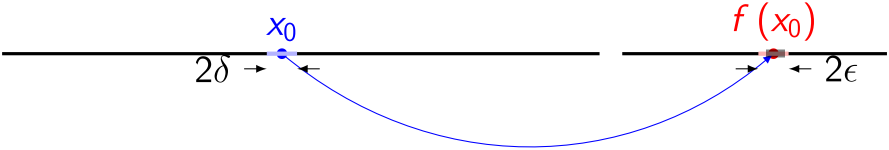

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Consider the interval $(f(x_0) - \epsilon, f(x_0) + \epsilon)$ of width 2ϵ centered around $f(x_0)$.
The definition of continuity means that we can always find a sufficiently small open interval centered at x_0 so that f carries it *inside* the interval $(f(x_0) - \epsilon, f(x_0) + \epsilon)$.

Limits on the real line

If $f : S \subset \mathbb{R} \to \mathbb{R}$ is defined in a neighbourhood of x_0, except possibly at x_0, then it has a limit a at $x_0 \in S$ if $\forall \epsilon > 0, \exists \delta > 0$:

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We write

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A function f is continuous at x_0 iff :

- $f(x_0)$ exists.
- $\lim_{x \longrightarrow x_0} f(x)$ exists.
- $\lim_{x \longrightarrow x_0} f(x) = f(x_0)$.

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The geometric viewpoint stresses the importance of open intervals, and hence, distance in the discussion of limits and continuity.

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Using the precise definition - an example

Theorem

If the functions $f : S \subset \mathbb{R} \to \mathbb{R}$ *and* $g : S \to \mathbb{R}$ *are both continuous at* $x = x_0 \in S$*, then so is the function* $f + g : S \to \mathbb{R}$.

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- Choose $\delta = \min\{\delta_1, \delta_2\}$.

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$\square$

- What did we use in this proof?

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□

- What did we use in this proof?
- The inequality :

$$|x - z| \leq |x - y| + |y - z|.$$

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The ordered pair (U, d) where d is a metric function defined on U is called a **metric space**.

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Given two metric spaces (U, d_U) and (V, d_V), a function $f : U \to V$ is defined to be continuous at $u_0 \in U$ if :$\forall \epsilon > 0, \; \exists \delta > 0 \; :$

$$d_U(u, u_0) < \delta \implies d_V(f(u), f(u_0)) < \epsilon.$$

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  to the pair of points $x = (x_1, x_2)$ and $y = (y_1, y_2)$.
- It satisfies all the properties of a distance function.

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A function $u \; : \; S \subset \mathbb{R}^2 \to \mathbb{R}, \; (x, y) \mapsto u(x, y)$ defined in some neighbourhood of the point (x_0, y_0), except possibly at (x_0, y_0), has the limit u_0 at (x_0, y_0) if $\forall \epsilon > 0, \; \exists \delta > 0$:

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$$\sqrt{(x-x_0)^2+(y-y_0)^2} < \delta \implies |u(x, y) - u\,(x_0, y_0)| < \epsilon$$

Examples

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$$|u(x,y) - 0| = r\left|\cos^3\theta\right| < \epsilon$$

whenever $0 < \sqrt{x^2+y^2} = r < \epsilon$.

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$$|u(x,y) - 0| = r\left|\cos^3\theta\right| < \epsilon$$

whenever $0 < \sqrt{x^2+y^2} = r < \epsilon$. Hence, taking $\delta = \epsilon$ allows the inequality in the definition of the limit to be satisfied. □

An important property

If the limit $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ exists, then $u(x,y)$ must approach u_0 as the point (x,y) approaches the point (x_0,y_0) along *any* curve.

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If we can find two curves C_1, C_2 going through (x_0,y_0) along which $u(x,y)$ approaches two different values u_1 and u_2 as (x,y) approaches (x_0,y_0), then $\lim_{(x,y)\to(x_0,y_0)} u(x,y)$ does not exist.

Examples

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \text{ does not exist}$$

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The limit $\lim_{(x,y)\to(0,0)} u(x,y)$ does not exist. □

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The open ball plays the role of the open interval!

Continuity on the complex plane

A function $f : D \subset \mathbb{C} \to \mathbb{C}$ is continuous at $z = z_0 \in D$ if $\forall \epsilon > 0, \exists \delta > 0$:

$$|z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon$$

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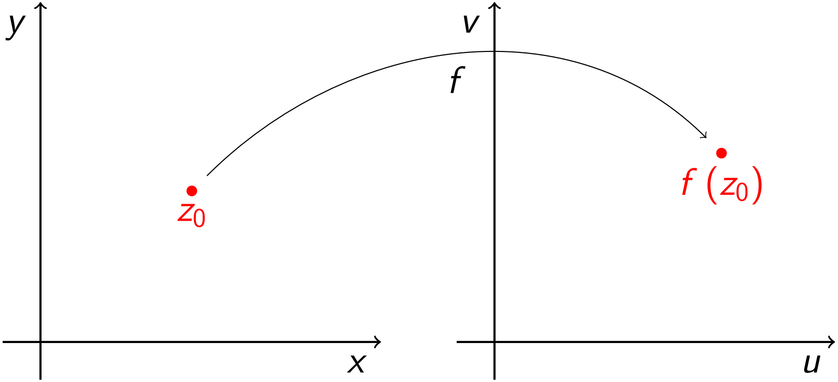
Geometrically, this means that given any open ϵ-ball centered at $f(z_0)$, $D_\epsilon(f(z_0))$, there exists an open δ-ball centered at z_0, $D_\delta(z_0)$, every point of which is carried by the map f *inside* $D_\epsilon(f(z_0))$:

$$f(D_\delta(z_0)) \subset D_\epsilon(f(z_0))$$

Continuity on the complex plane



f maps $z = x + \imath y$ to $w = u + \imath v$.

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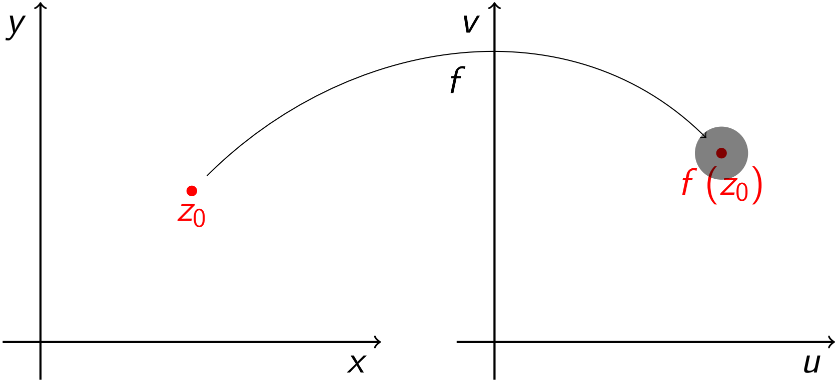

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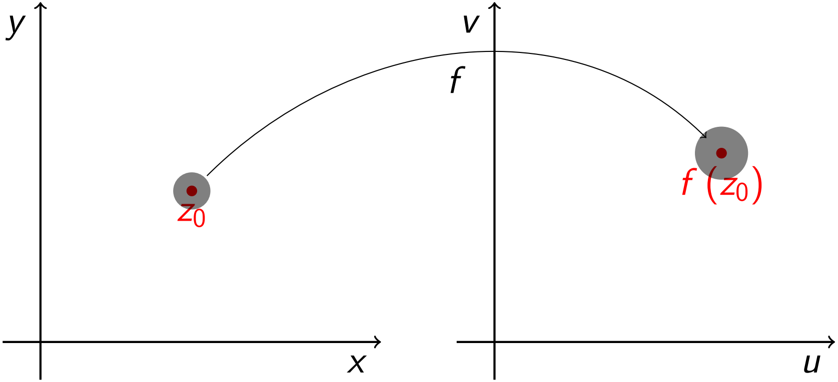

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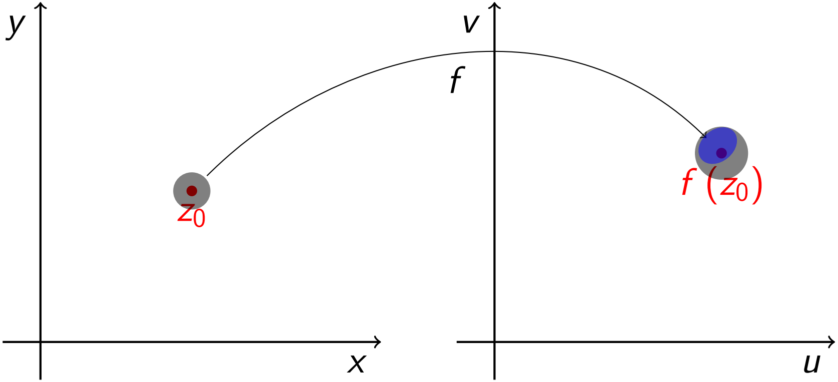

Given $D_1 = D_\epsilon(f(z_0))$, we can find an open disc $D_\delta(z_0)$ that is sufficiently small so that it maps entirely inside D_1.

Properties of continuity

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- Their quotient $\frac{f(z)}{g(z)}$ provided that $g(z_0) \neq 0$.
- Their composition $f(g(z))$ provided that $f(z)$ is continuous in a neighborhood of the point $g(z_0)$.

Limits of complex functions

A function $f : D \subset \mathbb{C} \to \mathbb{C}$ has a limit w_0 at $z = z_0 \in D$ if $\forall \epsilon > 0$, $\exists \delta > 0$:

$$0 < |z - z_0| < \delta \implies |f(z) - w_0| < \epsilon$$

We write

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Corollary : A function continuous at z_0 has a limit at z_0 and the value of the limit is $f(z_0)$.

Theorem

Let $f(z) = u(x,y) + \mathrm{i}v(x,y)$ *be a complex function that is defined in some neighbourhood of* z_0*, except perhaps at* $z_0 = x_0 + \mathrm{i}y_0$*. Then*

$$\lim_{z \longrightarrow z_0} f(z) = w_0 = u_0 + \mathrm{i}v_0$$

iff

$$\lim_{(x,y)\longrightarrow(x_0,y_0)} u(x,y) = u_0, \qquad \lim_{(x,y)\longrightarrow(x_0,y_0)} v(x,y) = v_0$$

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Thus, $0 < |z - z_0| < \delta \implies$

$$|u(x,y) - u_0| < \epsilon, \ |v(x,y) - v_0| < \epsilon$$

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$$|f(z) - w_0| \ \leq \ |u(x, y) - u_0| + |v(x, y) - v_0|$$

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Let $\lim_{z \to z_0} f(z) = A$ and $\lim_{z \to z_0} g(z) = B$. Then

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- $\lim_{z \to z_0} f(z)g(z) = AB$
- $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{A}{B}$, where $B \neq 0$.