Complex Functions
Differentition

Ananda Dasgupta

MA211, Lecture 7

Differentition in $\mathbb{R}$ and elsewhere

Recall that for a map $f : \mathbb{R} \to \mathbb{R}$, the derivative at $x_0 \in \mathbb{R}$ is defined by

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All of this works for $f : \mathbb{C} \to \mathbb{C}$!

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Very similar to what we did on $\mathbb{R}$!

An example :

$$f(z) = z^3$$

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- $\frac{d}{dz}[f(z)g(z)] = f'(z)g(z)+g'(z)f(z)$.
- $\frac{d}{dz}\frac{f(z)}{g(z)} = \frac{f'(z)g(z)-g'(z)f(z)}{[g(z)]^2}$, provide $g(z) \neq 0$.
- $\frac{d}{dz}f(g(z)) = f'(g(z))g'(z)$.

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- The condition for differentiability is more stringent on $\mathbb{C}$.

A non-differentiable function

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$f(z) = \bar{z}$ is differentiable nowhere!

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where

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What conditions must the functions $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfy for f to be differentiable?

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$$\begin{aligned}\Delta f = & \left[u\left(x_0 + \Delta x, y_0\right) + iv\left(x_0 + \Delta x, y_0\right)\right] \\ & - \left[u\left(x_0, y_0\right) + iv\left(x_0, y_0\right)\right]\end{aligned}$$

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$z \longrightarrow z_0$ along the real axis :

$$\frac{\Delta f}{\Delta z} = \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i\frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}$$

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$$\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + \mathfrak{i} \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}$$

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$$\begin{aligned} \Delta f = \; & [u\,(x_0, y_0 + \Delta y) + \mathrm{i}\,v\,(x_0, y_0 + \Delta y)] \\ & - [u\,(x_0, y_0) + \mathrm{i}\,v\,(x_0, y_0)] \end{aligned}$$

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$$\frac{\Delta f}{\Delta z} = \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\mathrm{i}\Delta y} + \mathrm{i}\frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\mathrm{i}\Delta y}$$

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$$\frac{\partial f}{\partial z} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

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Necesary condition for f to be differentiable :

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— the Cauchy-Riemann conditions!

Example

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The Cauchy-Riemann conditions are not satisfied anywhere!

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The Cauchy-Riemann conditions are not satisfied anywhere!

This function is differentiable nowhere.

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$f(z)$ is only differentiable on the X axis.

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- The question is, are they also **sufficient**?
- No!
- To see this, all that we neeed is a single counterexample!

Are the CR conditions sufficient ?

$$f(z) = \begin{cases} \frac{\bar{z}^2}{z} & \text{for } z \neq 0 \\ 0 & \text{for } z = 0 \end{cases}$$

$u(0,0) = 0$ and for $(x,y) \neq (0,0)$:

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$z \to 0$ along the Y axis leads to the same limit!

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f is **not** differentiable at the origin.

Sufficient conditions for differentiability

Let $f(z) = u(x, y) + \imath v(x, y)$ be a continuous function that is defined in some neighborhood of $z_0 = x_0 + \imath y_0$. If all the partial derivatives u_x, u_y, v_x and v_y are *continuous* at the point (x_0, y_0) and if the Cauchy-Riemann equations

$$u_x(x_0, y_0) = v_y(x_0, y_0), \qquad u_y(x_0, y_0) = -v_x(x_0, y_0)$$

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In this case, we can calculate the derivative using either

$$f'(z_0) = u_x(x_0, y_0) + \imath v_x(x_0, y_0)$$

or

$$f'(z_0) = v_y(x_0, y_0) - \imath u_y(x_0, y_0)$$

Example

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$$f'(z)$$

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