Holomorphic functions

Ananda Dasgupta

MA211, Lecture 8

Necessary and Sufficient conditions for differentiability

Let $f(z) = u(x, y) + iv(x, y)$ be a continuous function that is defined in some neighborhood of $z_0 = x_0 + iy_0$. If all the partial derivatives u_x, u_y, v_x and v_y are *continuous* at the point (x_0, y_0) and if the Cauchy-Riemann equations

$$u_x(x_0, y_0) = v_y(x_0, y_0), \qquad u_y(x_0, y_0) = -v_x(x_0, y_0)$$

hold, then $f(z)$ is differentiable at z_0.

Necessary and Sufficient conditions for differentiability

Let $f(z) = u(x, y) + \imath v(x, y)$ be a continuous function that is defined in some neighborhood of $z_0 = x_0 + \imath y_0$. If all the partial derivatives u_x, u_y, v_x and v_y are *continuous* at the point (x_0, y_0) and if the Cauchy-Riemann equations

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hold, then $f(z)$ is differentiable at z_0.

In this case, we can calculate the derivative using either

$$f'(z_0) = u_x(x_0, y_0) + \imath v_x(x_0, y_0)$$

or

$$f'(z_0) = v_y(x_0, y_0) - \imath u_y(x_0, y_0)$$

Example

$$f(z) = |z|^2 = z\bar{z}$$

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The derivative there is 0.
Let us verify this directly!

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The polar form of the Cauchy-Riemann equations

Let $f(z) = f\left(re^{i\theta}\right) = u(r,\theta) + iv(r,\theta)$ be a continuous function that is defined in some neighborhood of $z_0 = r_0e^{i\theta_0}$. If all the partial derivatives u_r, u_θ, v_r and v_θ are *continuous* at the point (r_0, θ_0) and if the Cauchy-Riemann equations

$$u_r\left(r_0,\theta_0\right) = \frac{1}{r_0}v_\theta\left(r_0,\theta_0\right), \quad v_r\left(r_0,\theta_0\right) = -\frac{1}{r_0}u_\theta\left(r_0,\theta_0\right)$$

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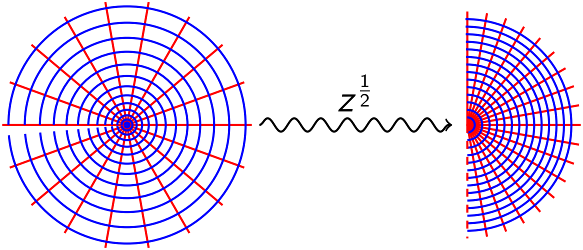
hold, then $f(z)$ is differentiable at z_0.
In this case, we can calculate the derivative using either $f'\left(z_0\right) = e^{-\mathrm{i}\theta_0}\left[u_r\left(r_0,\theta_0\right) + \mathrm{i}v_r\left(r_0, y\theta_0\right)\right]$
or $f'\left(z_0\right) = \frac{1}{r_0} e^{-\mathrm{i}\theta_0}\left[v_\theta\left(r_0,\theta_0\right) - \mathrm{i}u_\theta\left(r_0,\theta_0\right)\right]$

$f(z) = z^{\frac{1}{2}} = r^{\frac{1}{2}} \cos\frac{\theta}{2} + \mathfrak{i} r^{\frac{1}{2}} \sin\frac{\theta}{2}$ where the domain is restricted to be $r > 0$ and $-\pi < \theta \leq \pi$.



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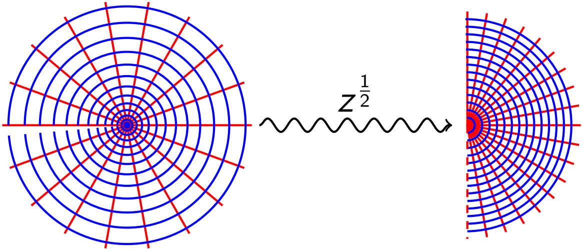

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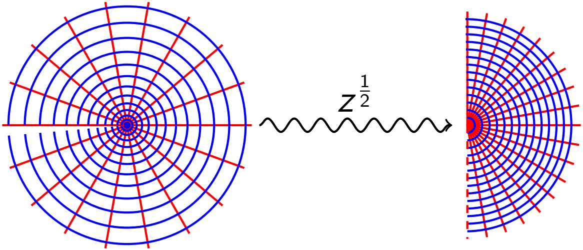

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It is differentiable everywhere else!

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The polar CR equations are satisfied.

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- However, the exact position of the branch cut is artificial.
- By changing the location of the branch, we can make the function differentiable everywhere
- except at the branch point - which is common to all branch cuts!

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• A function that is holomorphic over the entire complex plane is called **entire**.

Holomorphic a.k.a Analytic functions

- It is not very interesting to deal with functions that are differentiable at isolated points.
- A function $f : D \subset \mathbb{C} \to \mathbb{C}$ is said to be **holomorphic** at $z_0 \in D$ if it is differentiable at all points in some neighborhood containing z_0.
- A function is called holomorphic in D if it is holomorphic at all points in D.
- A function that is holomorphic over the entire complex plane is called **entire**.
- Points in $\mathbb{C}$ where f is not holomorphic are called **singular points**. These are of great importance in applications.

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- In complex analysis the two terms analytic and holomorphic are used interchangably.

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- If g is holomorphic at z_0 and f is holomorphic at $g(z_0)$, then $f \circ g \;:\; z \mapsto f(g(z_0))$ is holomorphic at z_0.

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- $z^{\frac{1}{2}}$ is holomorphic everywhere - except at the origin!

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Thus f is not holomorphic anywhere!

A non-analytic C^∞ real function

Consider the function :

$$f(x) = \begin{cases} \exp\left(-\frac{1}{x^2}\right) & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

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The Taylor expansion of $f(x)$ does not converge to $f(x)$!

This, despite the fact that it is differentiable to all orders everywhere!

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