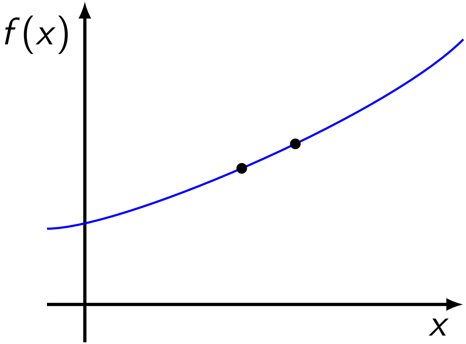
The Geometry of differentiation

Ananda Dasgupta

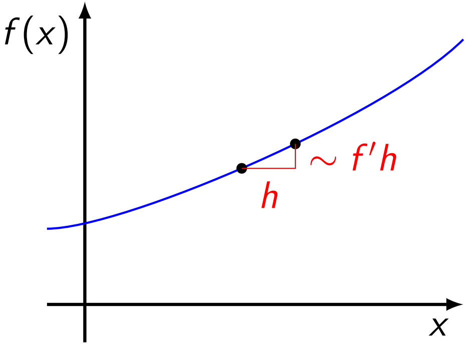
MA211, Lecture 9

The geometry of differentiation on $\mathbb{R}$



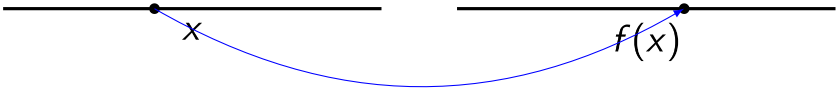
The geometric interpretation of the derivative of a real function $f : \mathbb{R} \to \mathbb{R}$ is that of the slope of its graph.

The geometry of differentiation on $\mathbb{R}$



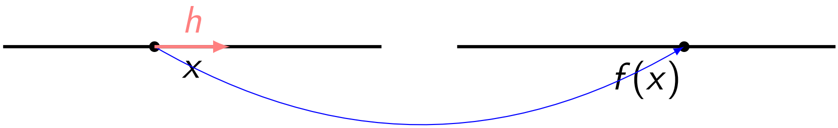
The geometric interpretation of the derivative of a real function $f : \mathbb{R} \to \mathbb{R}$ is that of the slope of its graph. For a small change h in x, the function $f(x)$ changes by $\approx f'(x)h$.

The geometry of differentiation on $\mathbb{R}$



Consider the map f as a way of carrying points of one copy of $\mathbb{R}$ to those in another copy of $\mathbb{R}$.

The geometry of differentiation on $\mathbb{R}$



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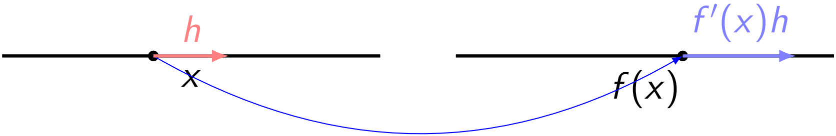
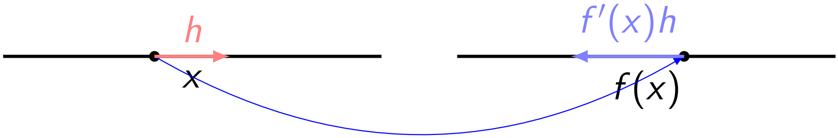
What does f do to a small vector of length h at x?

The geometry of differentiation on $\mathbb{R}$



Consider the map f as a way of carrying points of one copy of $\mathbb{R}$ to those in another copy of $\mathbb{R}$.
What does f do to a small vector of length h at x?
If gets mapped to another vector of length $|f'(x)|\,h$.

The geometry of differentiation on $\mathbb{R}$



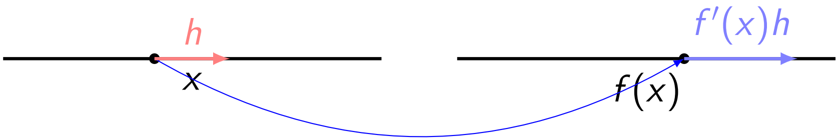
Consider the map f as a way of carrying points of one copy of $\mathbb{R}$ to those in another copy of $\mathbb{R}$.

What does f do to a small vector of length h at x?

If gets mapped to another vector of length $|f'(x)|\, h$.

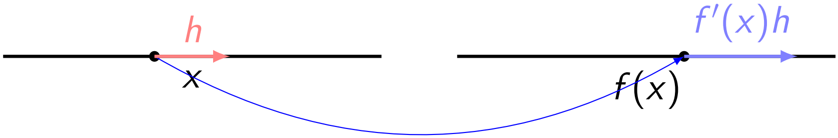
In addition, there will be a flip-over if $f'(x)$ is negative.

The geometry of differentiation on $\mathbb{R}$



The effect of f on small vectors at x is

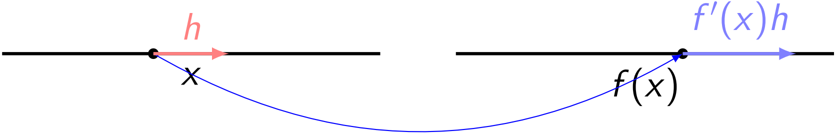
The geometry of differentiation on $\mathbb{R}$



The effect of f on small vectors at x is

- a scaling by $|f'(x)|$ and

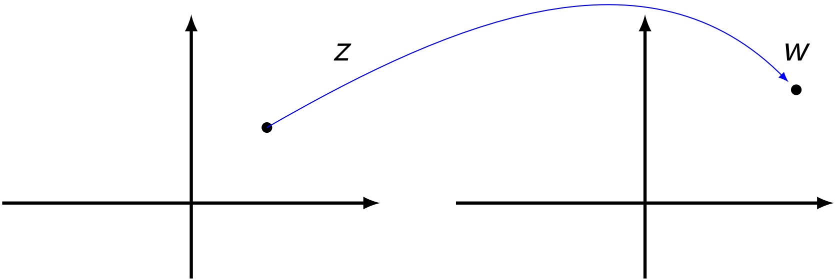
The geometry of differentiation on $\mathbb{R}$



The effect of f on small vectors at x is

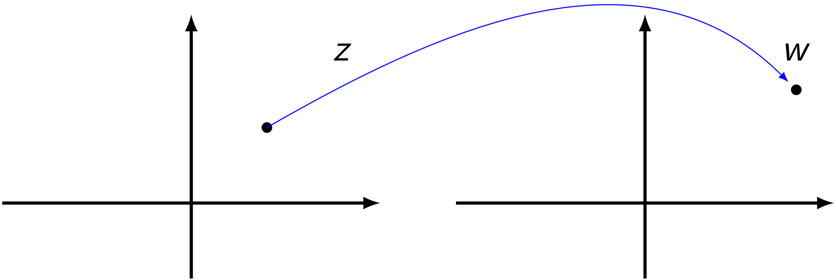
- a scaling by $|f'(x)|$ and
- a rotation by either 0 or π according as whether $f'(x)$ is positive or negative.

The geometry of differentiation on $\mathbb{C}$



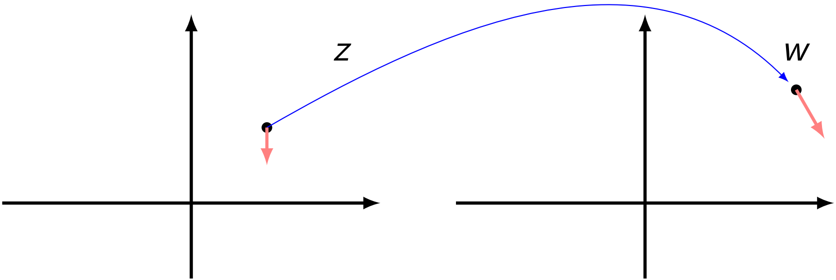
A map $f : \mathbb{C} \to \mathbb{C}$ carries points in one copy of the complex plane to another copy of the complex plane.

The geometry of differentiation on $\mathbb{C}$



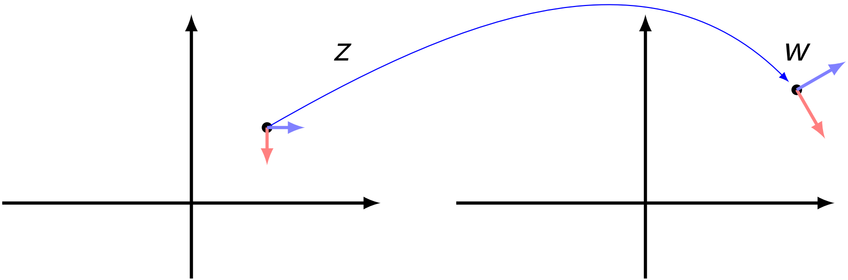
A map $f : \mathbb{C} \to \mathbb{C}$ carries points in one copy of the complex plane to another copy of the complex plane.
If f is differentiable at z, then it maps neighbouring points z, $z + \Delta z$ to $f(z)$ and $f(z + \Delta z) \approx f(z) + f'(z)\Delta z$, respectively.

The geometry of differentiation on $\mathbb{C}$



The infiniesimal vector Δz maps to $f'(z)\Delta z$.

The geometry of differentiation on $\mathbb{C}$

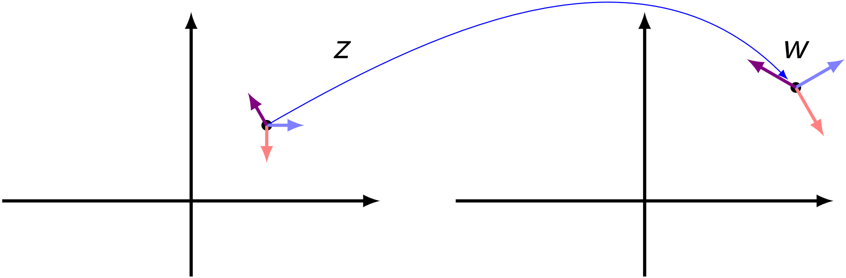


The infiniesimal vector Δz maps to $f'(z)\Delta z$.

Any small vector at z is

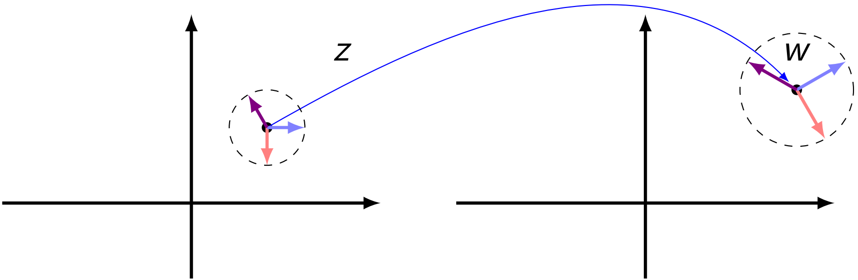
- scaled by a factor $|f'(z)|$
- rotated by $\arg(f'(z))$

The geometry of differentiation on $\mathbb{C}$



This makes a differentiable function $\mathbb{C} \to \mathbb{C}$ very different from a map $\mathbb{R}^2 \to \mathbb{R}^2$!

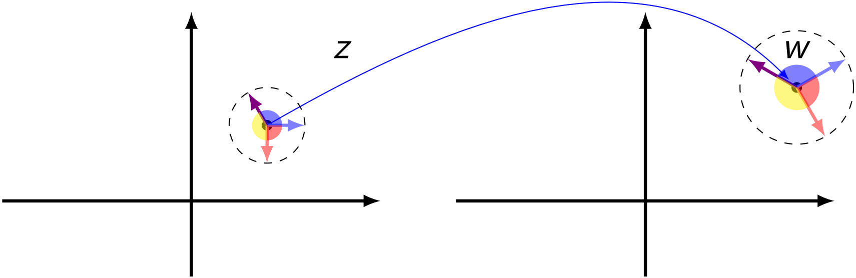
The geometry of differentiation on $\mathbb{C}$



This makes a differentiable function $\mathbb{C} \to \mathbb{C}$ very different from a map $\mathbb{R}^2 \to \mathbb{R}^2$!

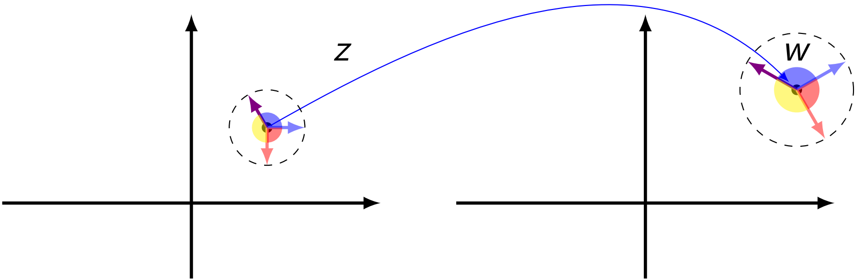
A small circle centered around z maps into a circle centered around $f(z)$.

The geometry of differentiation on $\mathbb{C}$



For a differentiable map

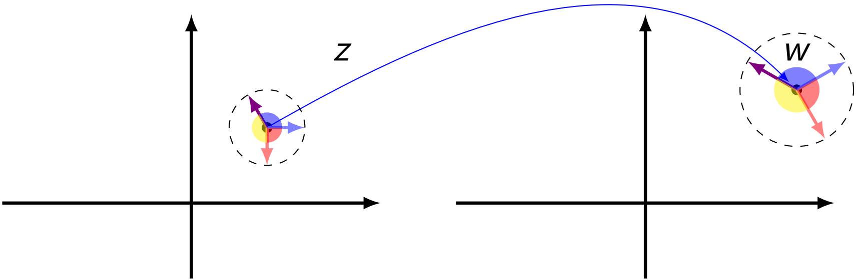
The geometry of differentiation on $\mathbb{C}$



For a differentiable map

- Angles between small vectors are preserved.

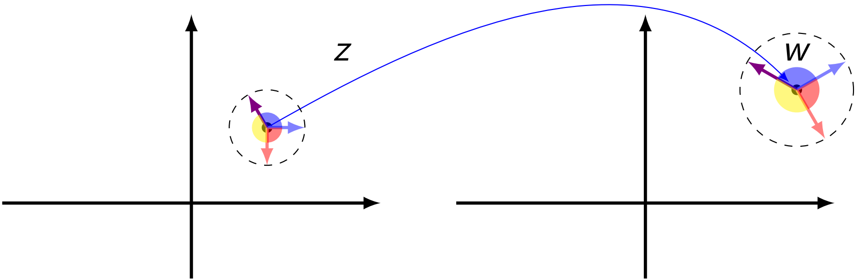
The geometry of differentiation on $\mathbb{C}$



For a differentiable map

- Angles between small vectors are preserved.
- All small vectors are scaled by the same factor.

The geometry of differentiation on $\mathbb{C}$



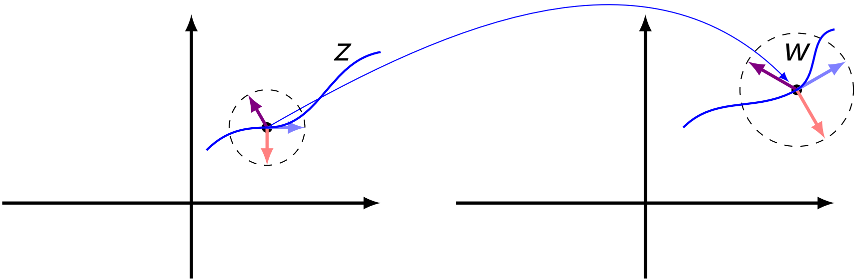
For a differentiable map

- Angles between small vectors are preserved.
- All small vectors are scaled by the same factor.

A differentiable map is *locally* a rotation cum a scaling!

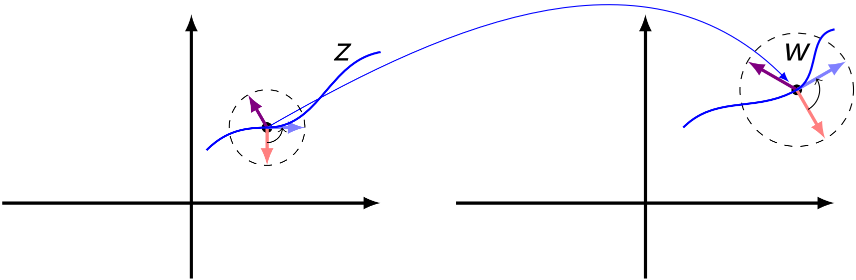
This is why small squares stay squares!

The geometry of differentiation on $\mathbb{C}$



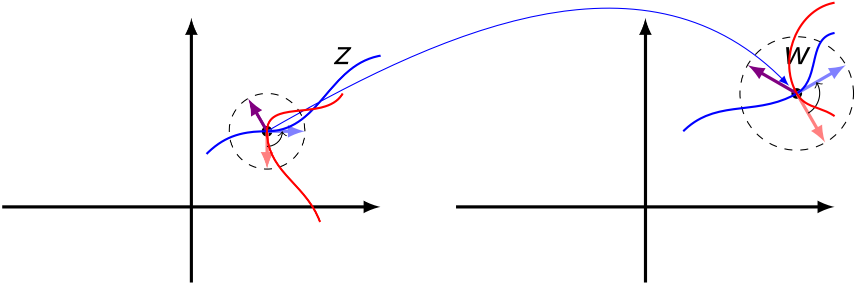
- Locally a curve looks like its tangent vector.

The geometry of differentiation on $\mathbb{C}$



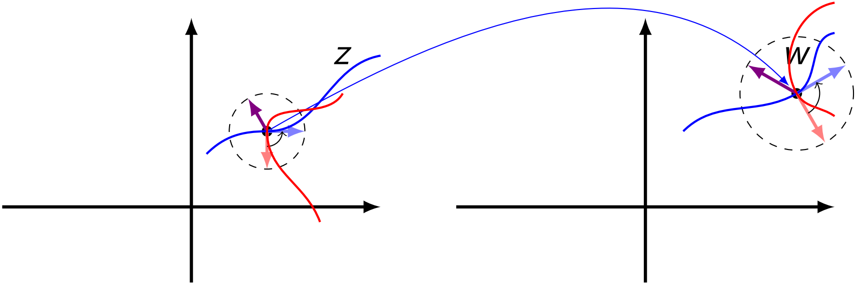
- Locally a curve looks like its tangent vector.
- Since a differentiable map preserves angles between small vectors,

The geometry of differentiation on $\mathbb{C}$



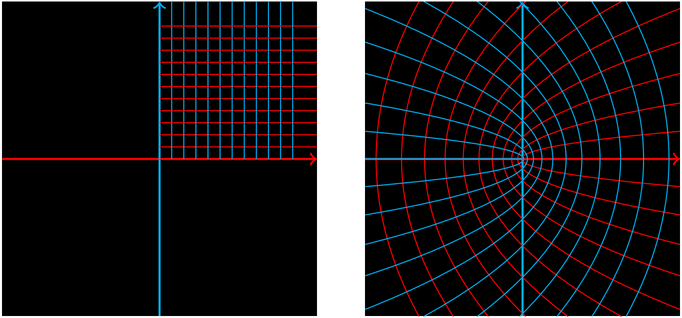
- Locally a curve looks like its tangent vector.
- Since a differentiable map preserves angles between small vectors,
- it preserves angles between curves!

The geometry of differentiation on $\mathbb{C}$



A differentiable map is **conformal**.

The geometry behind Cauchy-Riemann



A differentiable map is locally a rotation and a scaling.

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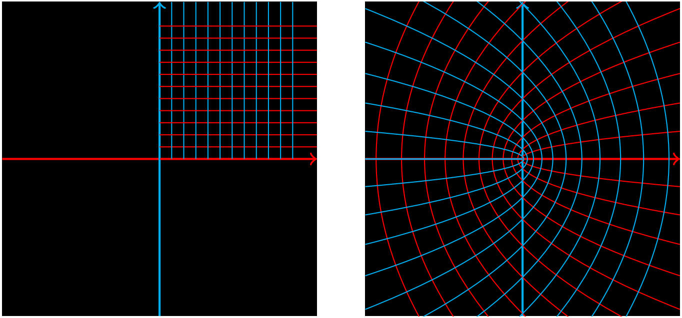
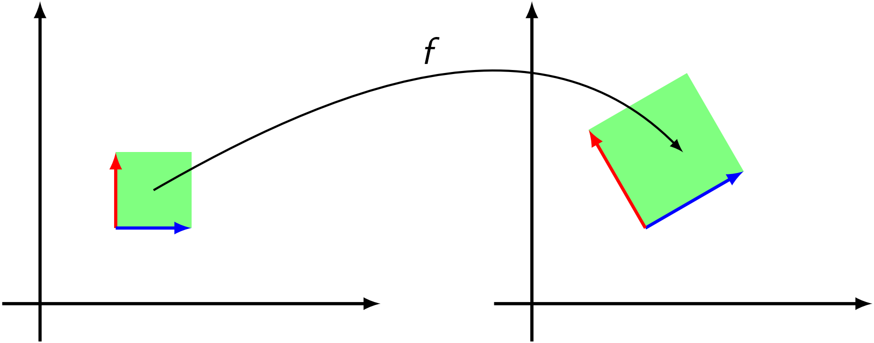


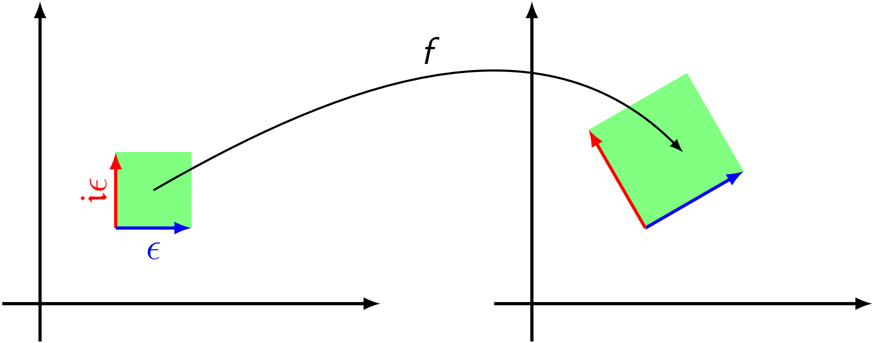
A differentiable map is locally a rotation and a scaling.
It maps "small" squares into "small" squares!

The geometry behind Cauchy-Riemann



Let us focus on one such small square.

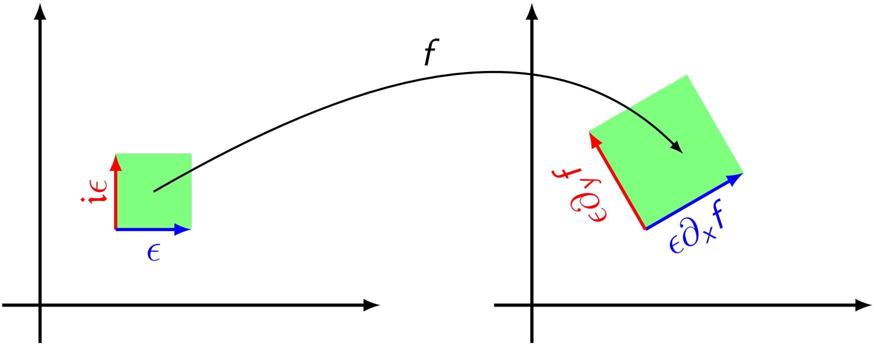
The geometry behind Cauchy-Riemann



Let us focus on one such small square.

$\Delta x = \Delta y = \epsilon$ means that the two adjacent sides are given by ϵ and $i\epsilon$.

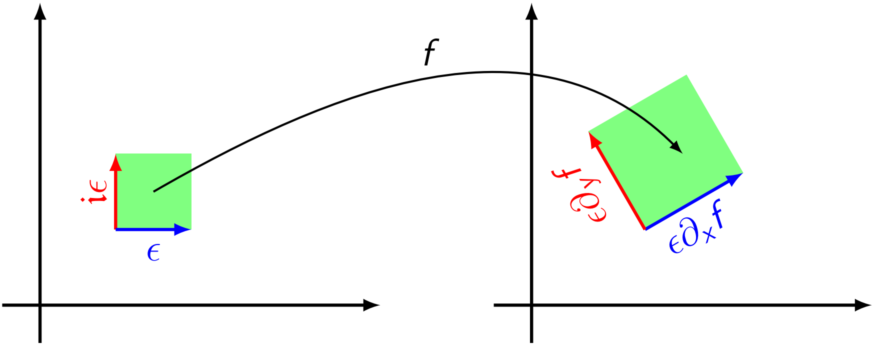
The geometry behind Cauchy-Riemann



As one moves a distance ϵ in the x direction, the change in f is $\epsilon\partial_x f$

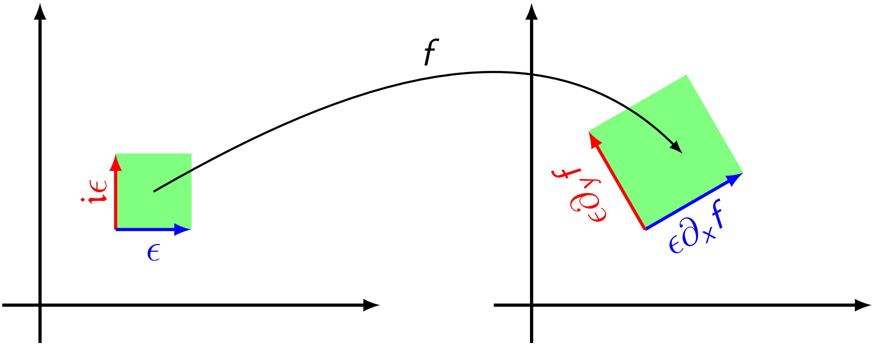
While for the y direction, the change is $\epsilon\partial_y f$

The geometry behind Cauchy-Riemann



$\epsilon\partial_x f$ and $\epsilon\partial_y f$ are two sides of a square.

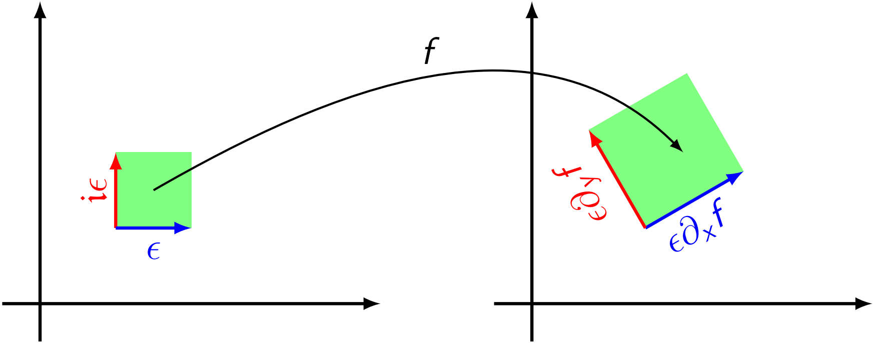
The geometry behind Cauchy-Riemann



$\epsilon\partial_x f$ and $\epsilon\partial_y f$ are two sides of a square.

They must be perpendicular!

The geometry behind Cauchy-Riemann



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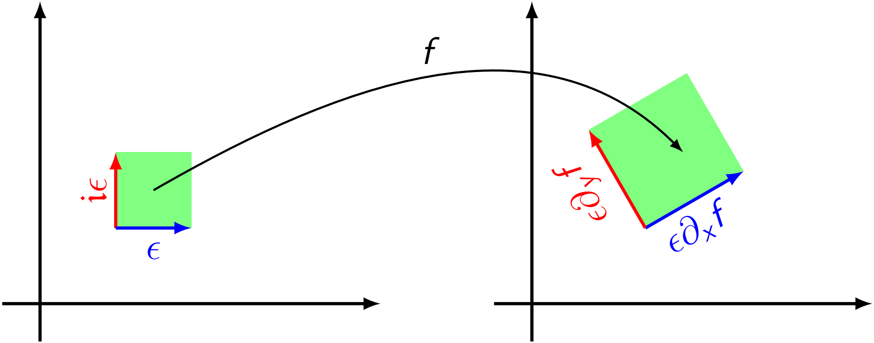
$$\mathrm{i}\epsilon\partial_x f = \epsilon\partial_y f$$

The geometry behind Cauchy-Riemann



$\epsilon\partial_x f$ and $\epsilon\partial_y f$ are two sides of a square.

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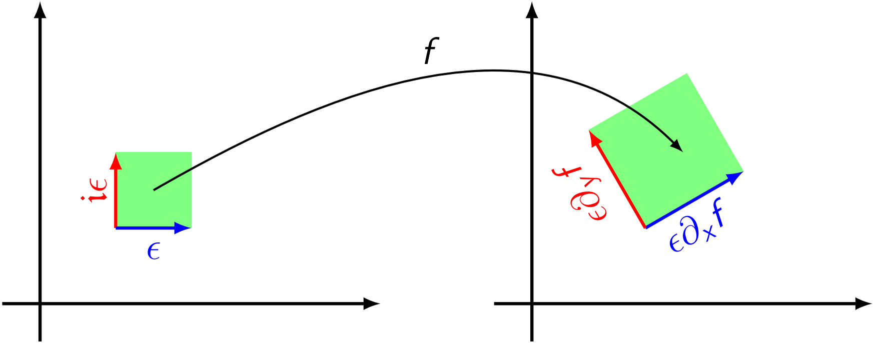
$$i\epsilon\partial_x(u + iv) = \epsilon\partial_y(u + iv)$$

The geometry behind Cauchy-Riemann



$\epsilon\partial_x f$ and $\epsilon\partial_y f$ are two sides of a square.

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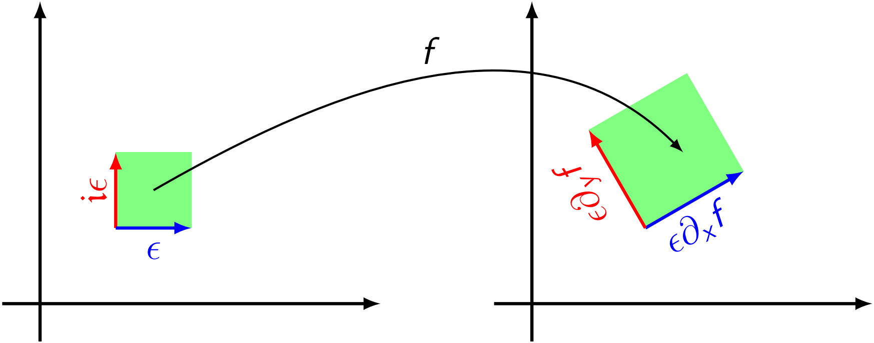
$$i\epsilon\partial_x(u + iv) = \epsilon\partial_y(u + iv)$$

Equate real and imaginary parts.

The geometry behind Cauchy-Riemann



$\epsilon\partial_x f$ and $\epsilon\partial_y f$ are two sides of a square.
They must be perpendicular!

$$u_x = v_y, \quad u_y = -v_x$$

This gives the Cauchy-Riemann equations!

CR $\implies$ conformal

A function $f : \mathbb{C} \to \mathbb{C}$ can be regarded as a map $\mathbb{R}^2 \to \mathbb{R}^2$;

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$$

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This maps a nearby point $(x+h, y+k)$ to

$$\begin{pmatrix} u(x+h,y+k) \\ v(x+h,y+k) \end{pmatrix} \approx \begin{pmatrix} u(x,y) + hu_x + ku_y \\ v(x,y) + hv_x + kv_y \end{pmatrix}$$

CR $\implies$ conformal

Thus f maps an infinitesimal vector $(h,k)^T$ to a new vector

$$\begin{pmatrix} hu_x + ku_y \\ hv_x + kv_y \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix}$$

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This, of course, works for any $\mathbb{R}^2 \to \mathbb{R}^2$ map.

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The matrix

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

is called the **Jacobian** matrix for the map.

CR $\implies$ conformal

If f is differentiable as a $\mathbb{C} \to \mathbb{C}$ map, then

$$u_x = v_y, \;\; u_y = -v_x$$

CR $\implies$ conformal

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Note that this is very similar to the rotation matrix

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

except that the determinant is wrong!

CR $\implies$ conformal

If f is differentiable as a $\mathbb{C} \to \mathbb{C}$ map, then

$$u_x = v_y, \; u_y = -v_x$$

Then the Jacobian becomes

$$\begin{pmatrix} u_x & u_y \\ -u_y & u_x \end{pmatrix} = \sqrt{u_x^2 + v_x^2} \begin{pmatrix} \frac{u_x}{\sqrt{u_x^2+v_x^2}} & \frac{u_y}{\sqrt{u_x^2+v_x^2}} \\ \frac{-u_y}{\sqrt{u_x^2+v_x^2}} & \frac{u_x}{\sqrt{u_x^2+v_x^2}} \end{pmatrix}$$

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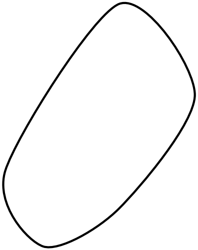
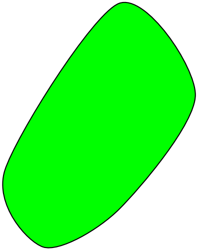
The CR equations lead to the Jacobian being a scaling cum a rotation - and thus conformal!

Holomorphy restricts!



- If a holomorphic function is known everywhere **on** a closed curve

Holomorphy restricts!



- If a holomorphic function is known everywhere **on** a closed curve
- It is known everywhere inside!