Liénard-Wiechert Potentials
(EM-III Seminar Writeup)

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The wave equation for the four-vector potential under the Lorentz gauge comes out to be

$$\Box A^{\mu} = \mu_0 j^{\mu} \tag{1}$$

where $j^{\mu}(\vec{r},t) = (\rho(\vec{r},t)c, \rho(\vec{r},t)\vec{v}(\vec{r},t))$

For a point charge (q) with given trajectory $\vec{w}(t)$

$$\rho(\vec{r},t) = q\delta^3(\vec{r} - \vec{w}(t))$$

and

$$\vec{v}(t) = \frac{d\vec{w}(t)}{dt}$$

$$\Rightarrow j^{\mu}(\vec{r},t) = (q\delta^3(\vec{r} - \vec{w}(t))c, q\delta^3(\vec{r} - \vec{w}(t))\vec{v}(t)) \tag{2}$$

$$\Rightarrow j^{\mu}(\vec{r},t) = \frac{q\delta^3(\vec{r} - \vec{w}(t))U^{\mu}(t)}{\gamma}$$

where $U^{\mu}(t) = \frac{d(ct,\vec{w}(t))}{d\tau} = (c\gamma, \vec{v}(t)\gamma)$

τ is the proper time in the particles intantaneous rest frame

and $\gamma = \frac{1}{\sqrt{1-\frac{v^2(t)}{c^2}}}$

$$\Rightarrow j^{\mu}(\vec{r},t) = \int \frac{q\delta^3(\vec{r} - \vec{w}(t'))\delta(t-t')U^{\mu}(t')}{\gamma} dt'$$

$$= qc \int \delta^3(\vec{r} - \vec{w}(t'))\delta(ct - ct')U^{\mu}(t')d\tau'$$

where we have used the following

$\delta(ct - ct') = \frac{\delta(t-t')}{c}$

$\gamma\Delta\tau' = \Delta t'$

$$\Rightarrow j^{\mu}(\vec{r},t) = qc \int d\tau' \delta^4(r - w(t'))U^{\mu}(t') \tag{3}$$

Now, let at time t (in the lab frame), time in the instantaneous rest frame of the particle be τ (the spatial coordinates of the particle in this frame is $(0,0,0)$). Then, we will always have a Poincaré transformation $({}^{\tau}\Lambda, {}^{\tau}a)$ relating the particles space-time coordinates in the instantaneous rest frame of the charged particle to that of the lab frame. In particular, the space-time coordinates of the charged particle in the two frames, $(ct, \vec{w}(t))$ in lab frame and $(c\tau, 0, 0, 0)$ in the particle's instantaneous rest frame, are related as follows

$$({}^{\tau}\Lambda^{\mu}_{0})c\tau + {}^{\tau} a^{\mu} = w^{\mu}(t)$$

$$\tilde{w}^{\mu}(\tau) \equiv ({}^{\tau}\Lambda^{\mu}_{0})c\tau + {}^{\tau} a^{\mu} \tag{4}$$

$$\Rightarrow U^{\mu}(t) = \frac{dw^{\mu}(t)}{d\tau} = \frac{d\tilde{w}^{\mu}(\tau)}{d\tau} \equiv \tilde{U}^{\mu}(\tau) \tag{5}$$

Using eqn(4) and eqn(5) in eqn (3) we get

$$\boxed{j^{\mu}(\vec{r},t) = qc\int d\tau' \delta^4(r - \tilde{w}(\tau'))\tilde{U}^{\mu}(\tau')} \tag{6}$$

Also, for eqn(1) we had obtained the retarded Green's function to be

$$G_r(r-r') = \frac{\delta(ct - ct' - R)}{4\pi R}\theta(t-t')$$

where $R = |\vec{r} - \vec{r'}|$

$$\Rightarrow G_r(r-r') = \frac{\delta(r^0 - r'^0 - R)}{4\pi R}\theta(r^0 - r'^0) \tag{7}$$

Now, since

$$\delta((r-r')^2) = \delta((r^0 - r'^0)^2 - R^2)$$

using

$$\delta(f(z)) = \sum_n \frac{\delta(z - z_n)}{\left|\frac{df(z)}{dz}\right|_{z=z_n}} \qquad \text{where } f(z_n) = 0 \tag{8}$$

$$\Rightarrow \delta((r-r')^2) = \frac{1}{2R}\left(\delta(r^0 - r'^0 - R) + \delta(r^0 - r'^0 + R)\right)$$

$$\Rightarrow \theta(r^0 - r'^0)\delta((r-r')^2) = \frac{1}{2R}\theta(r^0 - r'^0)\delta(r^0 - r'^0 - R) \tag{9}$$

Using eqn(9) in eqn(7) we get

$$\boxed{G_r(r-r') = \frac{1}{2\pi}\theta(r^0 - r'^0)\delta((r-r')^2)} \tag{10}$$

$$A^{\mu}(\vec{r},t) = \mu_0 \int j^{\mu}(\vec{r'},t')G_r(r-r')d^4r'$$

using eqn(10)

$$= \frac{\mu_0}{2\pi}\int j^{\mu}(\vec{r'},t')\theta(r^0 - r'^0)\delta((r-r')^2)d^4r'$$

Using eqn(6) here, we can get the $A^{\mu}(\vec{r},t)$ for a point charge (q) moving in a given trajectory (Liénard-Wiechert Potentials)

$$A^{\mu}(\vec{r},t) = \frac{\mu_0}{2\pi}qc\int d\tau' \int d^4r' \delta^4(r' - \tilde{w}(\tau'))\tilde{U}^{\mu}(\tau')\theta(r^0 - r'^0)\delta((r-r')^2)$$

$$= \frac{\mu_0}{2\pi}qc\int d\tau' \tilde{U}^{\mu}(\tau')\theta(r^0 - \tilde{w}^0(\tau'))\delta((r - \tilde{w}(\tau'))^2)$$

It can easily be argued using the postulates of special theory of relativity that $(r - \tilde{w}(\tau))^2$ has only one zero satisfying $(r^0 - \tilde{w}^0(\tau)) > 0$ Suppose the zero occurs when $\tau = \tau_0$, ie $r = \tilde{w}(\tau_0)$. Then using eqn(8) in the previous equation and integrating we obtain

$$\boxed{A^\mu(\vec{r}, t) = \frac{\mu_0}{4\pi} qc \left[\frac{\tilde{U}^\mu(\tau')}{\tilde{U}^\nu(\tau')[r - \tilde{w}(\tau)]_\nu} \right]_{\tau=\tau_0}} \tag{11}$$