Entanglement in a Desktop model for Hawking radiation: An Application of Quadratic Algebras

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Motivation

- Intradisciplinary links in physical systems important to understand cosmological objects.Since it is very difficult to actually test cosmological predictions , analogue laboratory models can serve as important tools in understanding cosmological phenomena.
- One such process is the remarkable prediction by Hawking that in the intense gravitational field around a black hole , quantum fluctuations create pairs of particles , which are split apart, one falling into the black hole and the other radiating away in such a manner that the black hole appears to be emitting thermal radiation at late times

Pair Creation Near the Event Horizon

Virtual pair creation close to the horizon can result in a real particle escaping to $\mathcal{I}^+$ which carries away a fraction of the black hole mass

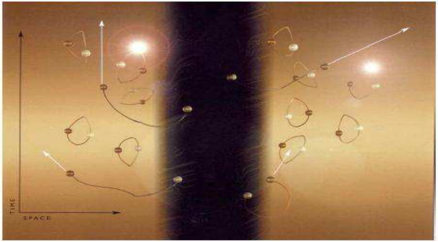


Figure: Hawking Effect(creation of particle pairs at the event-horizon.

Hawking Radiation and Information Loss

- Black holes emit Hawking radiation.Outgoing Hawking radiation is entangled with some in falling Hawking radiation carrying negative energy, reducing black hole mass.
- When a black hole evaporates, what happens to the matter that formed it or fell into it (Mathur)
- Black hole singularity projects to some maximally entangled state of matter and infalling radiation (Horowitz and Maldacena)
- Understanding possible through simple models where back reaction effects (quantum gravity effects) can be included .

Analogue models.

Desktop analogues for simulating Hawking radiation blossoming : new techniques in condensed matter physics and quantum optics.

- Acoustic models in moving fluids or BEC's where supersonic fluid (atomic) flow generates acoustic analogue of a black hole ,the the presence of phononic Hawking radiation from the acoustic horizon is derived.(Unruh)
- Picosecond pulsed lasers passing through glass, causing production of a disturbance zone in an optical black hole have been produced in the laboratory . This disturbance zone exhibits the emission of analogue hawking radiation for an "optical black hole".
- Fibre Optical Artificial White and Black Holes.(Leonhardt, Sivaprasad) These analogies work because Hawking Radiation comes due to existence of the Event Horizon, a one way membrane and Laboratory systems allow us to create equivalent one way systems.

Optical Black Holes

- An ultrashort laser pulse moving through **microstructured** optical fibre establishes a moving medium due to **"Kerr effect"** (nonlinear medium). When the velocities of both the medium and the probe light become equal then probe light gets trapped in between the pulses creating event-horizons.
- This is similar to trapping of a boat in a river near a waterfall (so velocity of river is changing)l, at a particular instant of time the velocity of the river becomes equal to the velocity of the boat.
- At that instant the boat gets trapped at a particular region i.e no boat can enter into that region and boat can escape from that inside region . The situation is similar to that of at the black hole and white hole.

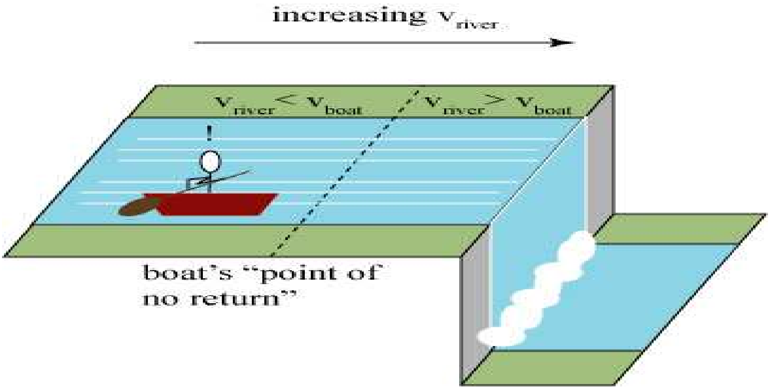


Figure: formation of event-horizons in the river.

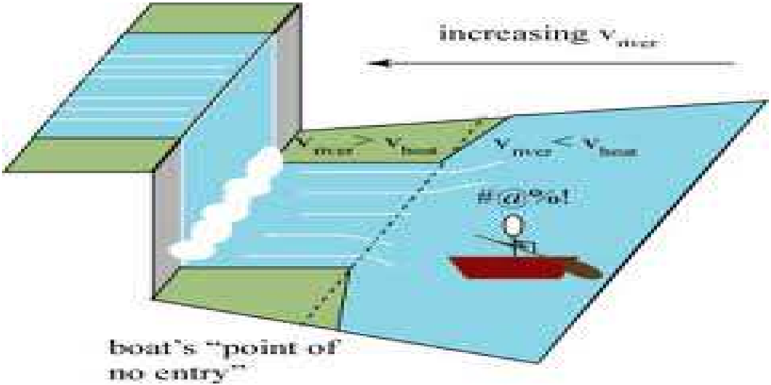


Figure: formation of event-horizons in the river.

- In an optical medium with variable refractive index (velocity), the velocity of the medium equal to the velocity of the probe light, the light gets trapped. So,there is a formation of Black hole and white hole in the fibre at the front end and the trailing end.
- At the horizon of an astrophysical black hole light freezes,reaching wavelengths shorter than the Planck scale.At the trailing end of the pulse the incoming probe modes are get compressed,oscillating with increasing frequency; they get blue shifted. at the black hole-horizon the reverse occurs; the probe light modes get elongated,oscillating with lesser frequency;they get red-shifted.

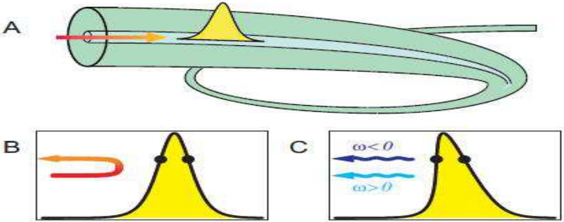


Figure 1: Fiber-optical horizons. A a light pulse in a fiber slows down infrared probe light attempting to overtake it. The diagrams below are in the co-moving frame of the pulse. B Classical horizons. The probe is slowed down by the pulse until its group velocity matches the pulse speed at the points indicated in the figure, establishing a white-hole horizon at the back and a black-hole horizon at the front of the pulse. The probe light is blue-shifted at the white hole until the optical dispersion releases it from the horizon. C Quantum pairs. Even if no probe light is incident, the horizon emits photon pairs corresponding to waves of positive frequencies from the outside of the horizon paired with waves at negative frequencies from beyond the horizon. An optical shock has steepened the pulse edge, increasing the luminosity of the white hole.

Figure: formation of Black hole horizon and white hole horizon.

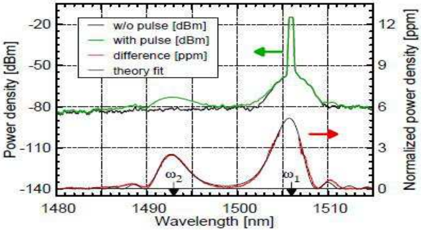


Figure 2: Measurement of blue shifting at a white-hole horizon. The black curve shows the power spectrum of probe light that has not interacted with the pulses, while the green curve displays the result of the interaction; both curves are represented on a logarithmic scale. The difference between the spectra on a linear scale, shown in red, exhibits a characteristic peak around the blue-shifted wavelength (ω_2) and another peak around the spectral line of the probe laser (ω_1) due to a gap in the probe light; both features indicate the presence of a horizon.

Figure: Blue and red shift of a probe light in the fibre.

Figure.(2) shows the difference in the spectrum of the probe light incident with ω_1 with and without the pulses, clearly displaying a blue-shifted peak at ω_2.

- We can quantitatively explain this effect by Doppler formula

$$\omega\prime = (1 - \frac{nu}{c})\omega$$

- In the near ultraviolet region, the medium moves with at superluminal speed.We see from the Doppler formula that these superluminal modes oscillate with negative frequencies $\omega\prime$ in the co=moving frame for positive frequenciesω in the laboratory frame, and vice versa.
- The pulse does not change $\omega\prime$,but it may partially convert sub- and superluminal partner modes into each other,thus creating Photons. Even if all the modes are initially in their vacuum states, the horizon spontaneously creates photon pairs.
- Photons with positive $\omega\prime$ correspond to the particles created at the outside of the Black hole, while the photons with negative frequency represent their partners beyond the horizon. This process represents the optical analogue of **"Hawking Radiation"**

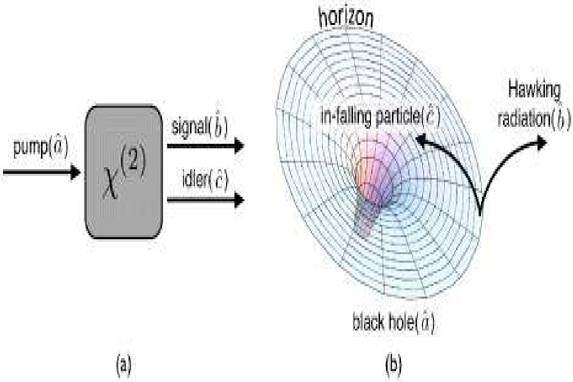
Our work

There are many other analogue models that may provide new light on perplexing theoretical questions concerning the information paradox problem. The information flow is in principle bi-directional and sometimes insights developed within the context of general relativity can be used to understand aspects of the analogue model. We seek to analyze a a model that keeps the the salient features with respect to just the information loss aspect and seeing whether the thermal nature of the radiation is modified when the black hole has evolved to a Planckian size and is therefore amenable to quantization. We consider a first quantized, zero dimensional model as an illustrative example in order to examine the structure of non-thermal contribution to the Hawking type radiation arising from the quantization of the black hole.

The analogy

- The two mode parametric amplifier with a bilinear Hamiltonian : traditionally used as a toy model for Hawking radiation.
- To include the effect of back reaction, consider the coupling of the gravitational field . Achieved by using a trilinear Hamiltonian.
- The outgoing modes :the signal mode, the incoming modes : the idler modes and the gravitational field of the black hole : the pump.
- IF pump mode is treated classically, signal mode (corresponds to the outgoing particle modes) and the idler mode(ingoing particle modes) form a two-mode squeezed state, and if we trace over the idler modes, the outgoing radiation can be mapped onto Hawking radiation. In the trilinear version, the pump mode is also quantized .

This , thus corresponds to the quantization of the black hole . Pictorially the analogy can be shown in figure 1.



Trilinear Hamiltonian

The Hamiltonian in the model system of interaction between the particle and black hole modes is given by (Nation and Blencoe)

$$\mathcal{H} = \omega_a \, a^\dagger a + \omega_b \, b^\dagger b + \omega_c \, c^\dagger c + \kappa \, a b^\dagger c^\dagger + \kappa^* \, a^\dagger b c \,. \tag{1}$$

Here a, b, and c are the boson annihilation operators of the quantized black hole modes , the incoming modes and the outgoing radiation modes respectively.

First, in order to show the entanglement between the modes b and c, we first treat the black hole mode *a* as classical (c number A) The Hamiltonian can be written in terms of the generators of the $SU(1,1)$ algebra

$$K_+ = b^\dagger c^\dagger \ , \qquad K_- = bc \ , \qquad K_0 = \frac{(b^\dagger b + c^\dagger c + 1)}{2}, \qquad (2)$$

as

$$\mathcal{H} = +\omega(2K_0 - 1) + \kappa A\, K_+ + \kappa^* A^* K_-. \qquad (3)$$

Entanglement

We can now find the entanglement properties as follows with $n_c - n_b = q$: The density matrix can be written as

$$\rho = \sum_{n_c, m_c} C_{n_c, n_c+q}\, C^*_{m_c+q, m_c}\, \big|n_c, n_c+q\big\rangle\big\langle m_c+q, m_c\big|\,, \tag{4}$$

$$C_{n_c, n_c+q} = \mathrm{sech}(A\kappa\tau)\sqrt{\frac{(n_c+q)!}{n_c!\; q!}}\; \mathrm{T}^{n_c}\,. \tag{5}$$

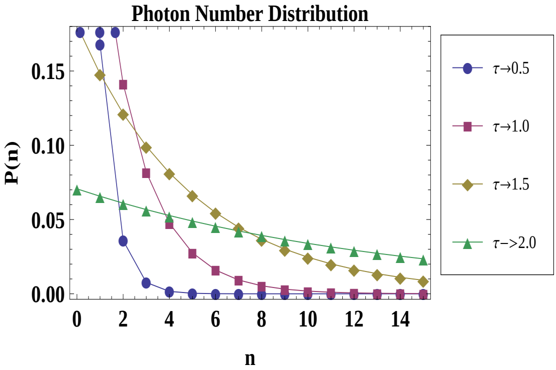
The partial transpose of ρ can be written as

$$\rho = \sum_{n_c, m_c} C_{n_c, m_c+q}\, C^*_{n_c+q, m_c}\, \big|n_c, m_c+q\big\rangle\big\langle n_c+q, m_c\big|\,, \tag{6}$$

For the special case $q = 0$ i.e $n_b = n_c = n$,

$$\rho_{PT} = \sum_{n,m} C_{n,m}\, C^*_{m,n}\, \big|n, m\big\rangle\big\langle m, n\big| \tag{7}$$

Where $C_{n,m} = \tanh^{m+n}(A\tau)\mathrm{sech}(A\kappa\tau)$.



Photon Number Distribution
P(n)
0.15
0.10
0.05
0.00
0
2
4
6
8
10
12
14
n
$\tau \to 0.5$
$\tau \to 1.0$
$\tau \to 1.5$
$\tau -> 2.0$

The eigenvalues are

$$\lambda_{nn} = \tanh^{2n}(A\tau)(sech(A\tau))/; n = m \quad (8)$$

$$\lambda_{nm} \pm \tanh^{m+n}(A\tau)(sech(A\tau))]n \neq m \quad (9)$$

Since the eigenvalues can assume negative values, this shows that the state is entangled. The Entropy of entanglement is given by

$$S = -Tr_n \rho log \rho = Cosh^2(A\kappa\tau) log_2(Cosh^2(A\kappa\tau)) - log_2(Sinh^2(A\kappa\tau))(1$$

As $\tau \longrightarrow \infty$ the states b and c are maximally entangled. To connect with the thermal nature of Hawking radiation we set $T = \frac{\hbar\omega_c}{2k_B ln(Coth(A\kappa\tau)}$

Quantized black-hole

In this we will show how one can handle quantized black-hole modes. The true symmetry of the system is then a quadratic algebra. This can be seen easily by Defining

$$Q_0 = \frac{1}{2}(K_0 - \hat{n}_a), Q_+ = aK_+, Q_- = a^\dagger K_-, \mathcal{L} = \frac{1}{2}(K_0 + a^\dagger a) \ . \tag{11}$$

The Hamiltonian Eq. (1) can be written in the form

$$\mathcal{H} = 2\omega_a \, \mathcal{L} + \kappa \, Q_+ + \kappa^* \, Q_- \ . \tag{12}$$

The operators thus defined satisfy the algebra

$$\begin{aligned} [Q_0, Q_\pm] &= \pm Q_\pm \\ [Q_+, Q_-] &= 3\, Q_0^2 + (2\mathcal{L} - 1)\, Q_0 - (\mathcal{K} + \mathcal{L}(\mathcal{L} + 1)). \end{aligned} \tag{13}$$

Which is a quadratic algebra, with the Casimir

$$\mathcal{C} = Q_+Q_- + Q_0^3 + (\mathcal{L} - 2)\, Q_0^2 - (\mathcal{K} + \mathcal{L}^2 + 2\mathcal{L} - 1)\, Q_0 + [\mathcal{K} + \mathcal{L}(\mathcal{L} + 1)].$$

Representation theory of quadratic algebra

$$\mathcal{L}\big|k,\ell,n_c\big\rangle = \ell\big|k,\ell,n_c\big\rangle$$
$$Q_0\big|k,\ell,n_c\big\rangle = (k+n_c-\ell)\big|k,\ell,n_c\big\rangle$$
$$Q_+\big|k,\ell,n_c\big\rangle = \sqrt{(n_c+1)(n_c+2k)(2\ell-k-n_c)}\big|k,\ell,n_c+1\big\rangle$$
$$Q_-\big|k,\ell,n_c\big\rangle = \sqrt{n_c(n_c+2k-1)(2\ell-k-n_c+1)}\big|k,\ell,n_c-1\big\rangle$$

With $n_c = 0,1,2,\cdots,(2\ell-k)$. For our purposes it is better to redefine the states in terms of the following constants, $n_b - n_c = q = 2k-1$ and $n_a + n_c = p - q = 2\ell - k$ so that $\big|k,\ell,n_c\big\rangle = \big|n_a,n_b,n_c\big\rangle = \big|p-q-n_c,n_c+q,n_c\big\rangle$.
But we are interested in the case where the signal modes are zero to begin with and all the energy is in the BH. This is the case where $q=0$ and $p\neq 0$, Setting these conditions in the above we get

Representation theory of quadratic algebra

$$Q_0|p-n_c,n_c,n_c\rangle = \left(\frac{4n_c-2p+1}{4}\right)|p-n_c,n_c,n_c\rangle$$

$$Q_+|p-n_c,n_c,n_c\rangle = \sqrt{(p-n_c)(n_c+1)(n_c+1)}\times |p-n_c-1,n_c+1,n_c+1\rangle$$

$$Q_-|p-n_c,n_c,n_c\rangle = \sqrt{(p-n_c+1)n_c\, n_c}|p-n_c+1,n_c-1,n_c-1\rangle$$

We will be using the above in the study of the time evolution of the full quantum TH.

Time Evolution

Let us denote the time evolution operator $U(t) = \exp(-i\mathcal{H}t)$, where $\mathcal{H}$ is given in Eq. (12). The fact that $\mathcal{L}$ is the central element of the quadratic algebra enable one to write the evolution operator in the form

$$U(t) = \exp(\tau\, Q_+ - \tau^*\, Q_-)\ \exp(-it2\omega_a\, \mathcal{L})\,. \tag{16}$$

With $\tau = -i\kappa t$ in the present case. Denoting an initial state by $|\Psi_0\rangle$. The action of $\exp(-it2\omega_a\, \mathcal{L})$ on this initial state gives only a phase. The interaction part of the Hamiltonian, as can be noticed gives rise to the PCS:

$$\big|\Psi, \tau\big\rangle = \frac{1}{\sqrt{N}}\ \exp(\tau\, Q_+ - \tau^*\, Q_-)\big|p, 0, 0\big\rangle\,. \tag{17}$$

Linearization of the algebra

Construct a new $\widetilde{Q}_-$ such that $[Q_+, \widetilde{Q}_-] = -2\lambda Q_0$ and $[Q_0, \widetilde{Q}_-] = -2\widetilde{Q}_-$. When $\lambda = \pm 1$ the algebra is either $su(1,1)$ or $su(2)$respectively. To find $\widetilde{Q}_-$ we choose an ansatz of the following form

$$\widetilde{Q}_- = F(\mathcal{C}, Q_0)\, Q_- . \tag{18}$$

It can be solved by substituting this ansatz in the linearized commutator,

$$\begin{aligned} Q_+\, F(\mathcal{C}, Q_0)\, Q_- - F(\mathcal{C}, Q_0)\, Q_-\, Q_+ &= -2\lambda Q_0 \\ F(\mathcal{C}, Q_0 - 1)\, Q_+\, Q_- - F(\mathcal{C}, Q_0)\, Q_-\, Q_+ &= -2\lambda Q_0 \,. \end{aligned} \tag{19}$$

In the present case it is

$$F(\mathcal{C}, Q_0) = \frac{Q_0(Q_0+1)\lambda + \epsilon}{\mathcal{C} - g(Q_0)}. \tag{20}$$

Where ϵ is an arbitrary constant that can be fixed by noting that the ground state of Q_- and $\widetilde{Q}_-$ is same. Note that in the present study $\lambda = 1$

Back to time evolution

$$\left|\Psi,\tau\right\rangle = \frac{1}{\sqrt{N}}\ \exp(\tau Q_{+} - \tau^{*}\widetilde{Q}_{-})\left|p,0,0\right\rangle . \qquad (21)$$

Where N is the time dependent normalization constant. Applying the disentanglement theorem to Eq. (21) we are left with

$$\left|\Psi,\mathrm{T}\right\rangle = \frac{1}{\sqrt{N}}\ \exp(\ \mathrm{T}\ Q_{+})\left|p,0,0\right\rangle . \qquad (22)$$

$\mathrm{T} = \tanh(\tau)$. The CS is found to be

$$\left|\Psi,\mathrm{T}\right\rangle = \frac{1}{\sqrt{N}}\sum_{n_c=0}^{p}\sqrt{\frac{\Gamma(p+1)}{\Gamma(p-n_c+1)}}\ \mathrm{T}^{n_c}\ \left|p-n_c,n_c,n_c\right\rangle , \qquad (23)$$

and the normalization constant

$$N = \frac{\exp\left[1/\mathrm{T}^2\right]\ \Gamma\left(p+1,1/\mathrm{T}^2\right)}{\mathrm{T}^{-2p}} . \qquad (24)$$

$\Gamma(a,z)$ is the incomplete gamma function

Entanglement properties

The quantum nature of the dynamics induced by the trilinear boson Hamiltonian leads to strong entanglement between the black hole and the particle-antiparticle subsystems This is a pure state and therefore the total entropy is zero. In our case S = 0, and hence the marginal entropies of the black hole and the particle antiparticle subsystems are equal: $S_a = S_{bc}$. Also, it follows from the Araki-Lieb theorem that $|S_b - S_c| \geq S_{bc} = S_a \geq |S_b + S_c|$. A quantitative measure of the entanglement between two subsystems is the index of correlation $I_{x-y} = S_x + S_y - S_{xy}$. In our case the index of correlation I_{a-bc}between the BH and Particle- antiparticle subsystems is equal to twice the marginal entropy of Black Hole:$I_{a-bc} = 2S_a$.

Denstity matrix

The density operator corresponding to the above abc state is

$$\rho_{abc} = \frac{1}{N}\sum_{n_c,m_c}^{p} \frac{\Gamma(p+1)\mathrm{T}^{n_c+m_c}\left|p-n_c,n_c,n_c\right\rangle\left\langle m_c,m_c,p-m_c\right|}{\sqrt{\Gamma(p-n_c+1)\Gamma(p-m_c+1)}}.$$

However,quantum correlations between the modes lead to the increase of the marginal entropy of each mode To see this consider the reduced denisty operator of the black-hole is obtained by performing the trace over the pair modes leading to

$$\begin{aligned}\rho_a = \frac{1}{N}\sum_{n_c=0}^{p} \frac{\Gamma(p+1)}{\Gamma(p-n_c+1)} \\ \mathrm{T}^{2n_c}\left|p-n_c\right\rangle\left\langle p-n_c\right| .\end{aligned} \tag{25}$$

Black-Hole modes

The photon-number distribution (PND) for the BH is

$$P(n_a) = \langle n_a | \rho_a | n_a \rangle = \frac{1}{N} \frac{\Gamma(p+1)}{\Gamma(n_a+1)} \mathrm{T}^{2p-2n_a} . \tag{26}$$

The entropy for the same is given by

$$S_a = -\sum_n P(n_a) \log[P(n_a)] \tag{27}$$

Outgoing Mode

To see the effective modification to the the hawking radiation with the incorporation of the quantum entanglement with the black hole modes For the outgoing particles the density matrix obtained by summing over the black hole and incoming modes is:

$$\rho_b = \frac{1}{N} \sum_{n_c=0}^{p} \frac{\Gamma(p+1)}{\Gamma(p-n_c+1)} \, \mathrm{T}^{2n_c} \, |n_c\rangle\langle n_c| \, . \tag{28}$$

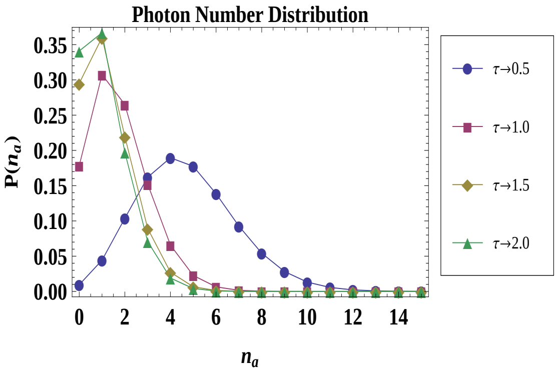
The PND for the signal is

$$P(n_b) = \langle n_b|\rho_b|n_b\rangle = \frac{1}{N} \frac{\Gamma(p+1)}{\Gamma(p-n_b+1)} \, \mathrm{T}^{2n_b} \, . \tag{29}$$

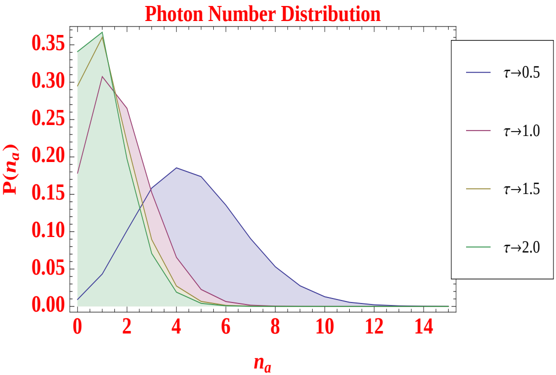
The entropy for the signal is

$$S_b = -\sum_n P(n_b) \, \log[P(n_b)] \tag{30}$$

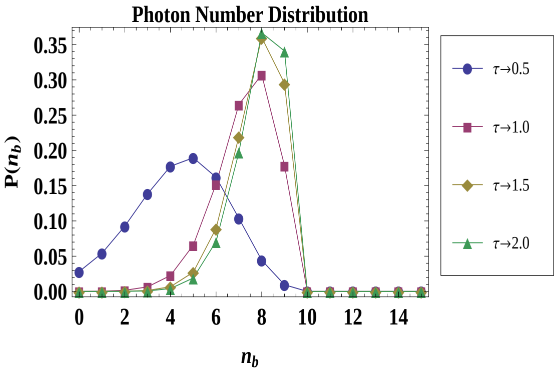
PND for Black-Hole modes



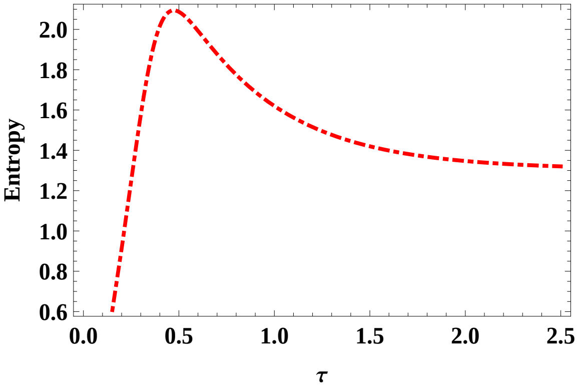
PND for Black-Hole modes



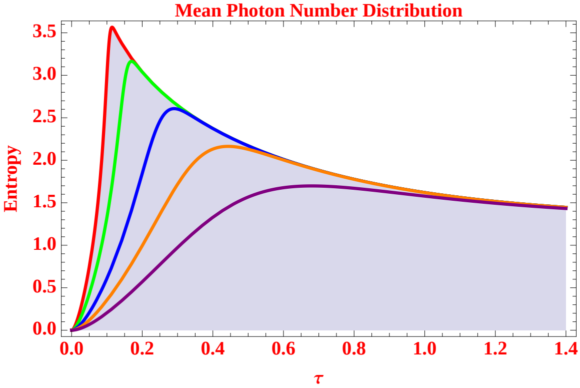
PND for outgoing modes



Entropy for Black-Hole modes

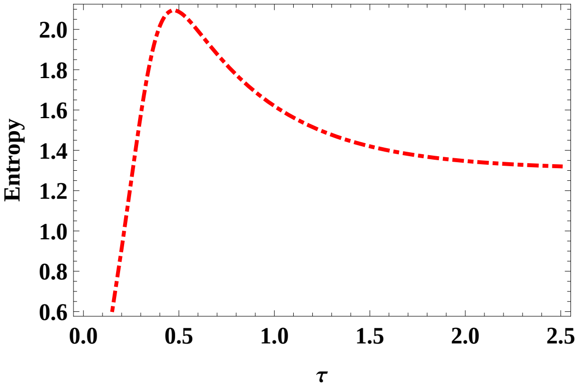


Entropy for Black-Hole modes

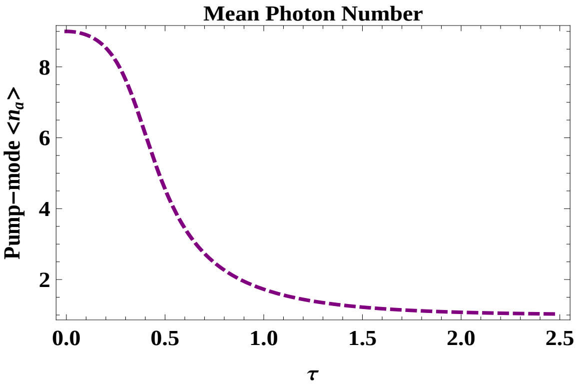


purple=p=5,orange=p=10,blue =p=20,green =p=50, red=p=100

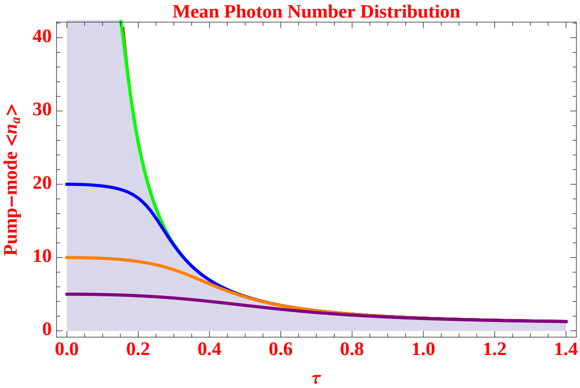
Entropy for outgoing modes as a function of time



Mean photon number for BH modes

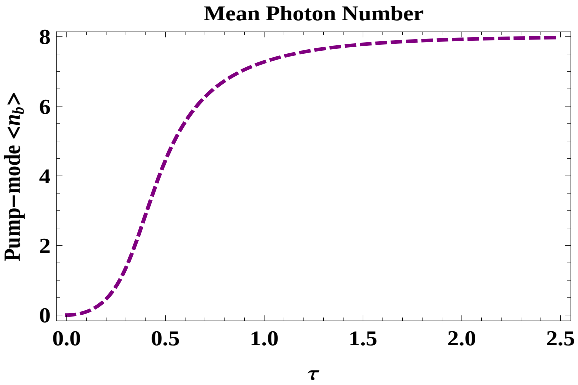


Mean photon number for BH modes



purple=p=5,orange=p=10,blue =p=20,green =p=50.

Mean photon number for outgoing modes



Super Radiance in Black holes and the Dicke Model analogy

The Hamiltonian of the Dicke model in the rotating wave approximation is

$$\mathcal{H}_{DM} = \omega_a \left(b^\dagger b + J_0\right) + \kappa \, bJ_+ + \kappa^* \, b^\dagger J_- \, . \tag{31}$$

Note that $J_0, J_\pm$ obey the usual angular momentum algebra. It is clear that the DM used to model Hawking radiation. In this case when a and c form an $SU(2)$ algebr

$$J_+ = a^\dagger c \, , \qquad J_- = ac^\dagger \, , \qquad J_0 = \frac{(a^\dagger c - c^\dagger a)}{2} \, . \tag{32}$$

The incoming modes and black hole form an atomic system, with the outgoing mode as interacting with the atomic system. In this way the incoming modes + Black hole encode information. This is the Dicke Model and shows superradiance which modifies the blackbody to grey body radiatyion. Work in progress.

The true underlying symmetry of the model can be discovered be defining

$$Q_0 = \frac{1}{2}(J_0 - \hat{n}_b), \quad Q_+ = b\, J_+, \quad Q_- = b^\dagger\, J_-, \quad \mathcal{L} = \frac{1}{2}(J_0 + \hat{n}_b)\,. \tag{33}$$

$$\mathcal{H} = \omega_b\, (b^\dagger b + J_0) + \kappa\, Q_+ + \kappa^* Q_-\,. \tag{34}$$

A straightforward calculation shows

$$[Q_0, Q_+] = Q_+, \quad [Q_0, Q_-] = Q_-, \tag{35}$$

$$[Q_+, Q_-] = -3Q_0^2 - (2\mathcal{L} + 1)Q_0 + [\mathcal{J} + \mathcal{L}(\mathcal{L} - 1)]\,. \tag{36}$$

Then the entanglement of the blackhole+incoming modes with the physical b modes (outgoing) modes can be calculated by constructing the coherent states of the deformed SU(2) algebra. It has been shown by various quantum optics people that the mean photon number rises and falls, a phenomenon called collapse and revival. This would be a truly dramatic signal. With the use of polynomial deformation of SU(2) algebras and constructing coherent sates we can show the effect of the total black-hole+ incoming radiation state entanled with the outgoing hawking radiation and look for effects of entanglement. Thus desktop optical and acoustic models can help us in understanding the black hole information paradox.

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