SYMMETRIES IN NON COMMUTATIVE QM

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NCQM

DOPLICHER, FRENHAGEN AND ROBERTS

$$[\hat{X}^{\mu}, \hat{X}^{\nu}] = i\Theta^{\mu\nu}$$

ALSO FROM STRING THEORY.
LANDAU PROBLEM

$$[P\hat{X}P, P\hat{Y}P] = \frac{1}{iB}$$

ALSO THROUGH BERRY-CURVATURE IN CERTAIN CONDENSED MATTER SYSTEMS

$$[X, Y] \sim i\Omega$$

Continued

$\{\Theta^{\mu\nu}\} \Rightarrow$ CONSTANT, NON-TENSOR.
$\Rightarrow$ LORENTZ SYMMETRY BROKEN IN QFT
USUAL METHODS $\longrightarrow$ BORROWED FROM PH.SP.Q

1.DEMOTE $\hat{X}^\mu \longrightarrow X^\mu$
2. USE STAR PRODUCT i.e. DEFORMED PRODUCT (MOYAL)

$$
\begin{aligned}
(f \star g)(x) &= f(x) e^{\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial_\mu}\overrightarrow{\partial_\nu}} g(x) \\
&= e^{\frac{i}{2}\theta^{\mu\nu}\partial^x_\mu \partial^y_\nu} f(x) g(y)\big|_{x=y}
\end{aligned}
$$

Continued

<u>A DIFFERENT NOTATION</u>

$$m^F(f(x) \otimes g(x)) = m(F^{-1}(f(x) \otimes g(x)))$$

Twist $\Rightarrow \mathcal{F} = e^{\frac{i}{2}\theta^{\mu\nu}P_\mu \otimes P_\nu} \in \mathcal{U}(P) \otimes \mathcal{U}(P)$
ALONG WITH MULTIPLICATION, THE CO-PRODUCT i.e. THE LEIBNITZ RULE TOO GETS DEFORMED

$$\begin{aligned} \triangle \rightarrow \triangle^F &= F \triangle F^{-1} \\ \triangle(M_{\mu\nu}) &= M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} \end{aligned}$$

Continued

AGAIN THE STAR PRODUCT IS NOT UNIQUE. FOR EXAMPLE, IN 2+1 DIM.(WITH $\theta^{0i} = 0$)

$$\Theta^{\mu\nu}_M = \begin{pmatrix} 0 & 0 & 0 \\ -0 & 0 & \theta \\ 0 & -\theta & 0 \end{pmatrix} ; \Theta^{\mu\nu}_V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -i\theta & \theta \\ 0 & -\theta & -i\theta \end{pmatrix}$$

M$\longrightarrow$ WEYL ORDERED; V$\longrightarrow$ NORMAL ORDERED

IT WAS BELIEVED THAT FIELD THEORIES CONSTRUCTED ON M/V $\star$-PRODUCT ARE EQUIVALENT AS

$$T(f \star_M g)(x) = (T(F) \star_V T(g))(x)$$

Continued

WHERE

$$T = e^{\frac{\theta}{4}\nabla^2}$$

RECENT CONTROVERSIES

PERSISTANCE OF EQIVALENCE AT THE LEVEL OF INTERACTING QFT

Continued

WE ADDRESS THE ISSUE AT THE LEVEL OF NCQM. $2D$ REPRESENTATION OF NC HEISENBERG ALGEBRA

$$\hat{X}_i : \psi(\hat{x}_i) \rightarrow \hat{x}_i \psi(\hat{x}_i)$$

$$\hat{P}_i : \psi(\hat{x}_i) \rightarrow \frac{\hbar}{\theta}\epsilon_{ij}[\hat{x}_j, \psi(\hat{x}_i)] = \hat{P}_i \psi$$

$\hat{b} = \frac{\hat{X}_1 + i\hat{X}_2}{\sqrt{2\theta}}$ s.t. $[\hat{b}, \hat{b}^\dagger] = 1$

<u>CLASS. HILBERT SPACE</u>

$\mathcal{H}_c = Span_c\{|n\rangle\}_{n=0}^{\infty}$;$|n\rangle = \frac{(\hat{b}^\dagger)^n}{\sqrt{n!}}|0\rangle$

Continued

QUANTUM HILBERT SPACE

$\mathcal{H}_Q =$ SPACE OF HILBERT-SCHMIDT OP. i.e. THE TRACE CLASS BOUNDED SET OF OPERATORS IN $\mathcal{H}_c$.

INNER PRODUCT

$(\psi|\phi) = tr_{\mathcal{H}_c}(\psi^\dagger \phi)$

VOROS BASIS AND VOROS WAVE FUNCTION

$\psi(z,\bar{z}) = \langle z|\psi(\hat{x},\hat{y})|z\rangle = tr(|z)\psi)$

$|\vec{x})_V \equiv |z) = |z\rangle\langle z|; \; |z\rangle = e^{-\bar{z}b + zb^\dagger}|0\rangle \in \mathcal{H}_c$

$z = \frac{X_1 + iX_2}{\sqrt{2\theta}}$

$|z\rangle \longrightarrow$ COHERENT STATE

Continued

PROPERTIES OF $|z)$

$(z|z') = e^{-|z-z'|^2}$; $\int \frac{d^2z}{\pi}|z) \star_V (z| = 1_Q$

$|\vec{p}) = \sqrt{\frac{\theta}{2\pi}} e^{i\vec{p}.\hat{\vec{x}}}; \hat{P}_i|\vec{p}) = p_i|\vec{p})$

s.t.

$\int d^2p|\vec{p})(\vec{p}| = 1$; $(\vec{p}|\vec{p'}) = \delta^2(\vec{p} - \vec{p'})$

THEN

$$|\vec{x})_V = \sqrt{\frac{\theta}{2\pi}} \int d^2p e^{-\frac{\theta}{4}\vec{p}^2} e^{-i\vec{p}.\vec{x}}|\vec{p})$$

Continued

MOYAL BASIS

$$|\vec{x}) = \int \frac{d^2p}{2\pi} e^{-i\vec{p}.\vec{x}} |\vec{p}) = \sqrt{\frac{\theta}{2\pi}} \int \frac{d^2p}{2\pi} e^{i\vec{p}.(\hat{\vec{x}}-\vec{x})}$$

<u>THEY SATISFY</u>

(i) $\int d^2x |\vec{x})_M \star_M {}_M(\vec{x}| = \int d^2x |\vec{x})_M {}_M(\vec{x}| = 1$

(ii)$(\vec{p}|\vec{x})_M = \frac{1}{2\pi} e^{-i\vec{p}.\vec{x}}$

(iii)${}_M(\vec{x'}|\vec{x})_M = \delta^2(\vec{x} - \vec{x'}) \rightarrow$ ORTHOGONAL

UNLIKE VOROS

(iv)${}_V(\vec{x'}|\vec{x})_M = \sqrt{\frac{2}{\pi\theta}} e^{-\frac{1}{\theta}(\vec{x}-\vec{x'})^2}$

Continued

INTRODUCE

$\hat{X}_i^c\psi = \frac{1}{2}(\hat{x}_i\psi + \psi\hat{x}_i) \Rightarrow [\hat{X}_i^c, \hat{X}_j^c] = 0$

EQUIVALENTLY:

$\hat{X}_i^c = \hat{x}_i + \frac{\theta}{2}\epsilon_{ij}\hat{P}_j$

THEY ADMIT COMMON EIGENSTATE. INDEED

$\hat{X}_i^c|\vec{x})_M = x_i|\vec{x})_M$

IMPOSE THE ADDITIONAL STRUCTURE OF AN ALGEBRA ON $\mathcal{H}_Q$

$\mu(|\psi) \otimes |\phi)) = |\psi\phi)$

Continued

<u>QUESTIONS</u>

WHAT IS THE FORM OF THE REPRESENTATION OF THIS PRODUCT STATE IN MOYAL OR VOROS BASIS AND,IN PARTICULAR, IS THERE A COMPOSITION RULE IN TERMS OF THE REPRESENTATIONS OF THE INDIVIDUAL STATES IN THESE BASES?

Continued

TAKE $|\psi) = \sqrt{\frac{\theta}{2\pi}} \int \frac{d^2p}{2\pi} \psi(\vec{p}) e^{i\vec{p}.\vec{x}}$

NOTE: NORMALIZABILITY

$\Rightarrow (\psi|\psi) = tr_c(\psi^\dagger \psi) < \infty$

$\Rightarrow \psi(\vec{p})$ IS SQUARE INTEGRABLE.

THEN

$_M(\vec{x}|\psi\phi) = \sqrt{2\pi\theta}_M(\vec{x}|\psi) \star_M {}_M(\vec{x}|\phi)$

$_V(\vec{x}|\psi\phi) = 4\pi^2_V(\vec{x}|\psi) \star_V {}_V(\vec{x}|\phi)$

WHERE $_M(\vec{x}|\psi) = \int \frac{d^2p}{(2\pi)^2} \psi(\vec{p}) e^{i\vec{p}.\vec{x}}$

$_V(\vec{x}|\psi) = \sqrt{\frac{\theta}{2\pi}} \int \frac{d^2p}{(2\pi)^2} \psi(\vec{p}) e^{-\frac{\theta}{4}\vec{p}^2} e^{i\vec{p}.\vec{x}}$

$= \sqrt{\frac{\theta}{2\pi}} e^{\frac{\theta}{4}\nabla^2} {}_M(\vec{x}|\psi)$

Continued

<u>OBSERVATIONS</u>

(i) M/V COMPOSITIONS ARE RELATED TO M/V BASES RESPECTIVELY.

(ii) ASSOCIATIVITY IS OBVIOUS.

(iii) $T = e^{\frac{\theta}{4}\nabla^2}$ OPERATOR RELATING M/V WAVE FUNCTIONS IS NON-UNITARY AND NON-INVERTIBLE.

Continued

INDEED VOROS WAVE-FUNCTIONS BELONG TO A SMALLER SUBSPACE ($\subset$ SCHWARTZ CLASS) UNLIKE THE MOYAL WAVE FUNCTIONS; HERE WE ALSO REQUIRE SMOOTHNESS$\sim \sqrt{\theta}$, AS MODES WITH HIGH MOMENTA ARE AUTOMATICALLY SUPPRESSED.
IT MAY HAVE EFFECTS AT THE PATH-INTEGRAL LEVEL.

Continued

(i) $\hat{X}_i^c$ CAN NOT BE A PHYSICAL OBSERVABLE, AS IT VIOLATE SPACE-SPACE UNCERTAINTY.
THUS A SYSTEM CAN NOT BE PREPARED IN MOYAL BASIS.
CORRESPONDINGLY, $\vec{x}$ IN $|\vec{x})_M$ CANNOT REFER TO POSITION $\vec{x}$.
(ii) VOROS BASIS IS IN CONFORMITY WITH UNCERTAINTY RELATION.
MUST WEAKEN VON-NEUMANN'S PVM$\longrightarrow$ POVM.

Continued

INDEED, $\Pi_x = \frac{1}{\pi}|\vec{x})_v \star_v {}_v(\vec{x}|$ FORM POVM i.e THEY ARE POSITIVE AND INTEGRATE TO IDENTITY.

PROBABILITY $P(\vec{x}) = tr_Q(\Pi_x \rho)$

FOR $\rho = |\psi)(\psi|$,

$P(\vec{x}) = (\psi|\vec{x})_v \star_v {}_v(\vec{x}|\psi)$

THIS POVM FAILS IN THE MOYAL CASE; POSITIVITY IS NOT SATISFIED.

Continued

TRANSITION AMPLITUDE IN VOROS BASIS

$$ {}_v(\vec{x}_f, T|\vec{x}_i, 0)_v = \frac{m}{m\theta + iT} e^{-\frac{m(\vec{x}_f - \vec{x}_i)^2}{2(m\theta + iT)}} $$

IN MOYAL BASIS, THE θ-DEPENDENCE DOES NOT OCCUR.
1. ALTHOUGH $\exists$ A FORMAL MATHEMATICAL EQUIVALENCE, IT IS THE VOROS BASIS, WHICH CAN BE CONSIDERED PHYSICAL
2. THIS SUGGESTS THAT AN ABSTRACT, BASIS INDEPENDENT FORMALISM SHOULD BE DEVELOPED IN $D = 3$.

Continued

$3D$ GENERALIZATION

Start With

$$[\hat{x}_i, \hat{x}_j] = i\theta_{ij} = i\epsilon_{ijk}\theta_k$$

$\Theta = \theta_{ij}$ is degenerate

$\Rightarrow$ it is possible to orient $\vec{\theta}$ along 3rd axis by an SO(3) rotation

i.e. $\hat{x}_i \rightarrow \hat{\bar{x}}_i = \bar{R}_{ij}\hat{x}_j$

s.t $[\hat{\bar{x}}_1, \hat{\bar{x}}_2] = i\theta, [\hat{\bar{x}}_1, \hat{\bar{x}}_3] = [\hat{\bar{x}}_2, \hat{\bar{x}}_3] = 0$

with $\hat{\bar{x}}_3$ being commutative

Continued

A Non-unique form of $\bar{R}$:

$$\bar{R} = \begin{pmatrix} \cos\alpha\cos\beta & \sin\beta\cos\alpha & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{pmatrix};$$

$$\vec{\theta} = \theta \begin{pmatrix} \sin\alpha\cos\beta \\ \sin\alpha\sin\beta \\ \cos\alpha \end{pmatrix}$$

$$\bar{\Theta} = \bar{R}\Theta\bar{R}^T = \begin{pmatrix} 0 & \theta & 0 \\ -\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Continued

CLASSICAL HILBERT SPACE

$\mathcal{H}_c^{(3)} = span\{|n, \bar{x}_3\rangle\} = span\{|z, \bar{x}_3\rangle\}$

Action of $\hat{x}_i$

$\hat{x}_i|n, \bar{x}_3\rangle = (\bar{R}^{-1})_{ij}\hat{\bar{x}}_j|n, \bar{x}_3\rangle =$

$(\bar{R}^{-1})_{i\alpha}\hat{\bar{x}}_\alpha|n, \bar{x}_3\rangle + (\bar{R}^{-1})_{i3}\bar{x}_3|n, \bar{x}_3\rangle$

$\alpha = 1, 2$

QUANTUM HILBERT SPACE

$$\begin{aligned}\mathcal{H}_q^{(3)} &= \{\psi(\hat{\bar{x}}_i) : tr_c\psi^\dagger\psi < \infty\} \\ &= \{\psi(\hat{\bar{x}}_i) : \int \frac{d\bar{x}_3}{\sqrt{\theta}} tr'_c\psi^\dagger\psi < \infty\}\end{aligned}$$

Continued

Here, $tr'_c \rightarrow$ restricted trace over the noncommutative $2D$ plane.

$\mathcal{H}_q^{(3)}$ is therefore simply a one-parameter family of $\mathcal{H}_q^{(2)}$.

Since the elements of $\mathcal{H}_q^{(3)}$ leaves the subspace $span\{|n, \bar{x}_3\rangle\} \subset \mathcal{H}_c^{(3)}$ (for fixed $\bar{x}_3$) invariant, one can also write

$$\mathcal{H}_q^{(3)} = \{\psi : [\bar{x}_3, \psi] = 0; tr_c \psi^\dagger \psi < \infty\}$$

Continued

ACTION OF MOMENTUM

Introduce $\hat{\bar{x}}_4$, such that $[\hat{\bar{x}}_j, \hat{\bar{x}}_4] = i\theta\delta_{j3}; j = 1, 2, 3$

Formally $\hat{\bar{x}}_4 = -i\theta\frac{\partial}{\partial\bar{x}_3}$ Then

$$\hat{\bar{P}}_\alpha\psi = \frac{1}{\theta}\Gamma_{\alpha\beta}[\hat{\bar{x}}_\beta, \psi]; \alpha, \beta = 1, 2, 3, 4$$

$$\Gamma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Note

1. $\hat{\bar{P}}_4\psi = 0$

Thus$\exists$ only three non-trivial momenta

Continued

POSITION OPERATOR

$\hat{X}_i : |\psi) \longrightarrow \hat{X}_i|\psi) = |\hat{x}_i\psi)$

Then NQ Heisenberg algebra is

$[\hat{X}_i, \hat{X}_i] = i\theta_{ij}; [\hat{X}_i, \hat{P}_j] = i\delta_{ij}; [\hat{P}_i, \hat{P}_j] = 0$

NORMALIZED MOMENTUM EIGENSTATES

$|\vec{p}) = \frac{\theta^{\frac{3}{4}}}{2\pi} e^{ip_i\hat{x}_i}$

Satisfies

(i) $\hat{\vec{P}}|\vec{p}) = \vec{p}|\vec{p})$

(ii) $(\vec{p'}|\vec{p}) = \delta^3(\vec{p'} - \vec{p})$

(iii) $\int d^3p|\vec{p})(\vec{p}| = 1_q$

Continued

MOYAL AND VOROS BASIS IN $3D$

$$\hat{X}_i^{(l)}\psi \equiv \hat{x}_i\psi$$

$$\hat{X}_i^{(r)}\psi \equiv \psi\hat{x}_i$$

$$\hat{X}_i^{(c)}\psi \equiv \frac{1}{2}(\hat{X}_i^{(l)} + \hat{X}_i^{(r)})\psi$$

By splitting $(\hat{x}_i\psi)$ into symmetric and anti-symmetric parts

$$\hat{X}_i^{(l)}\psi = \hat{X}_i^{(c)}\psi + \frac{1}{2}[\hat{x}_i, \psi]$$

Going back and forth between barred and un-barred frame yields

$$\hat{X}_i^{(c)} = \hat{X}_i^{(l)} + \frac{1}{2}\theta_{ij}P_j$$

s.t. $[\hat{X}_i^{(c)}, \hat{X}_j^{(c)}] = 0$

Continued

As in $2D$, here too, one can introduce Moyal/Voros basis

$$|\vec{x})_M = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} e^{-i\vec{p}.\vec{x}} |\vec{p})$$

$$|\vec{x})_V = \frac{\theta^{\frac{3}{4}}}{\sqrt{2\pi}} \int d^3p e^{-\frac{\theta}{4}\vec{p}^2} e^{-i\vec{p}.\vec{x}} |\vec{p})$$

Satisfying

$$_M(\vec{x}|\psi\phi) = 2\pi\theta^{\frac{3}{4}}{}_M(\vec{x}|\psi) \star_M {}_M(\vec{x}|\phi)$$

$$_V(\vec{x}|\psi\phi) = {}_V(\vec{x}|\psi) \star_V {}_V(\vec{x}|\phi)$$

With

$$|\psi) = \frac{\theta^{\frac{3}{4}}}{2\pi} \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \psi(\vec{p}) e^{-i\vec{p}.\hat{\vec{x}}}$$

Continued

ANGULAR MOMENTUM AND DEFORMED COPRO

Take $|\psi) = \int d^3p \psi(\vec{p}) e^{i\vec{p}.\hat{\vec{x}}} = \psi(\hat{\vec{x}})$

Scalar property

$|\psi^R) = U(R)|\psi) = \psi^R(\hat{\vec{x}}) = \psi(R^{-1}\hat{\vec{x}})$

$=\int d^3p \psi(\vec{p}) e^{i\vec{p}.(R^{-1}\hat{\vec{x}})}$

For $R = 1 + i\vec{\phi}.\vec{L}, |\vec{\phi}| \ll 1; (L_i)_{jk} = -i\epsilon_{ijk}; [L_i, L_j] = i\epsilon_{ijk}L_k$

$\psi^R(\hat{x}_i) = \psi(\hat{x}_i) + i\phi_i \hat{J}_i \psi(\hat{x}_i)$

$\hat{J}_i = \epsilon_{ijk}\hat{X}^c_j \hat{P}_k; [\hat{J}_i, \hat{J}_j] = i\epsilon_{ijk}\hat{J}_k$

$U(R) = e^{i\vec{\phi}.\vec{J}} \rightarrow$ The unitary representation of R in $\mathcal{H}_q$

Continued

ON DEFORMED LEIBNIZ RULE

Note $\hat{J}_i(\phi\psi) = \epsilon_{ijk}\hat{X}^c_i\hat{P}_k(\phi\psi)$

$\neq (\epsilon_{ijk}\hat{X}^c_j(\hat{P}_k\phi))\psi + \phi(\epsilon_{ijk}\hat{X}^c_j(\hat{P}_k\psi))$

Rather, $\hat{J}_i(\phi\psi) = \frac{1}{2}[\hat{x}_j\hat{P}_k(\phi\psi) + (\hat{P}_k(\phi\psi))\hat{x}_j]$

$= (\hat{J}_i\phi)\psi + \phi(\hat{J}_i\psi) + \frac{1}{2}[(\hat{P}_i\phi)((\vec{\theta}.\vec{P})\psi - ((\vec{\theta}.\vec{P})\phi)(\hat{P}_i\psi)]$

Thus the co-product is deformed:

$$\Delta_\theta(\hat{J}_i) = \Delta_0(\hat{J}_i) + \frac{1}{2}[\hat{P}_i \otimes (\vec{\theta}.\hat{\vec{P}}) - (\vec{\theta}.\hat{\vec{P}}) \otimes \hat{P}_j]$$

$$\Delta_0(\hat{J}_i) = \hat{J}_i \otimes 1 + 1 \otimes \hat{J}_i$$

Continued

DEFORMED CO-PRODUCT AND AUTOMORPHISM

$|\psi) \rightarrow |\psi^R) = \int d^3p\psi(\vec{p})e^{i\vec{p}.(R^{-1}\hat{\vec{x}})}$

$= \int d^3p\psi(\vec{p})e^{i(R\vec{p}).\hat{\vec{x}}}$

Like-wise for $|\phi)$

Then $|\psi\phi) = \int d^3pd^3p'\psi(\vec{p})\phi(\vec{p'})e^{i(\vec{p}+\vec{p'}).\hat{\vec{x}}}e^{-\frac{i}{2}\theta_{ij}p_ip'_j}$

But now $|(\phi\phi)^R) \neq |\psi^R\phi^R) =$

$m[\Delta_0(R)(|\psi) \otimes |\phi))]; \Delta_0(R) = R \otimes R$

Rather,$|(\psi\phi)^R) = U(R)[m(|\psi) \otimes |\phi)]$

$= m[\Delta_\theta(|\psi) \otimes |\phi))]$

Where $\Delta_\theta(R) = F\Delta_0(R)F^{-1} \rightarrow$ Def. co-prod.

$\Rightarrow F = e^{\frac{i}{2}\hat{P}_i \otimes \hat{P}_j}$

Continued

Thus the automorphism symmetry can be restored iff the deformed co-product is used.
Also, $\Delta_\theta(\hat{J}_i) = F\Delta_0(\hat{J}_i)F^{-1}$
In the multi-particle setting, the restoration of the automorphism symmetry relevant at the level of action.
Consider Schrodinger action
$S = \int dt \mathcal{L}$

$$\mathcal{L} = tr_c \psi^\dagger (i\partial_t - \frac{\hat{\vec{P}}^2}{2m} - V(\hat{x}_i))\psi$$

Continued

The $SO(3)$ symmetry is manifest iff $\mathcal{L}$ transform as a scalar,i.e.
$\psi^\dagger\psi \longrightarrow (\psi^\dagger\psi)^R; (\psi^\dagger V\psi) \longrightarrow (\psi^\dagger V\psi)^R$
Since, $tr_c(A)^R = tr_c A$ (We show it later)
For a Generic composite $'A'$ of fields.

Continued

ON THE CONSTANCY OF Θ

Rotated coordinate

$\hat{x}_i^R \equiv (R\hat{\vec{x}})_i = R_{ij}\hat{x}_j$

Then the commutator of the rotated coordinate

$[\hat{x}_i^R, \hat{x}_j^R] = \hat{x}_i^R\hat{x}_j^R - \hat{x}_j^R\hat{x}_i^R = i(R\Theta R^T)_{ij} = i(\Theta_{UD}^R)_{ij}$

$\Theta \longrightarrow \Theta_{UD} \longrightarrow$ 2nd rank antisymmetric tensor

Continued

Note $[\hat{x}_i^R, \hat{x}_j^R] = m[\Delta_0(R)(\hat{x}_i \otimes \hat{x}_j - \hat{x}_j \otimes \hat{x}_i)]$

But rotated commutator is

$([\hat{x}_i, \hat{x}_j])^R = (\hat{x}_i\hat{x}_j)^R - (\hat{x}_j\hat{x}_i)^R$

$= m[\Delta_\theta(R)(\hat{x}_i \otimes \hat{x}_j - \hat{x}_j \otimes \hat{x}_i)] = i\theta_{ij} = i(\Theta_D)_{ij}$

Here $(\hat{x}_i\hat{x}_j)^R = m[\Delta_\theta(R)(\hat{x}_i \otimes \hat{x}_j)]$

$= \hat{x}_i^R\hat{x}_j^R + \frac{i}{2}\theta_{ij} - \frac{i}{2}(\Theta_{UD}^R)_{ij}$

$\Rightarrow$ No longer a second rank tensor.

Also $\hat{J}_i\theta_{jk} = -i\hat{J}_i[m(\hat{x}_j \otimes \hat{x}_k - \hat{x}_k \otimes \hat{x}_j)]$

$= -im[\Delta_\theta(\hat{J}_i)(\hat{x}_j \otimes \hat{x}_k - \hat{x}_k \otimes \hat{x}_j)] = 0$

Continued

In a more general setting, the quantum position op.

$\hat{X}_i^{(l)}|\psi(\hat{x}_i)) = |\hat{x}_i\psi(\hat{x}_i)) = m(\hat{x}_i \otimes \psi(\hat{x}_i))$

And its rotated counterpart

$\hat{X}_i^{(l)R}|\psi(\hat{x}_i)) = R_{ij}\hat{X}_j^{(l)}|\psi(\hat{x}_i))$

Now under rotation

$m(\hat{x}_i \otimes \psi(\hat{x}_i)) \rightarrow U(R)[m(\hat{x}_i \otimes \psi(\hat{x}_i))]$

$= m[\Delta_\theta(R)(\hat{x}_i \otimes \psi(\hat{x}_i))] = \hat{\tilde{X}}_i^{(l)R}|\psi^R(\hat{x}_i))$

Here $\hat{\tilde{X}}_i^{(l)R} \equiv \hat{X}_i^{(l)R} + \frac{1}{2}[R, \Theta]_{ij}\hat{P}_j$

is the effective rotated quantum position op.

transform non-covariantly

Continued

But $[\hat{\tilde{X}}_i^{(l)R}, \hat{\tilde{X}}_j^{(l)R}] = i\theta_{ij} \longrightarrow$ again constant

<u>OBSERVATIONS</u>

The distinction between $\hat{X}_i^{(l)R}$ and $\hat{\tilde{X}}_i^{(l)R}$

Disappear in $D = 2$, as $[R, \Theta] = 0$

$\hat{\tilde{X}}_i^{(r)R} = \hat{X}_i^{(r)R} - \frac{1}{2}[R, \Theta]_{ij}\hat{P}_j$

$\hat{X}_i^{(c)} \longrightarrow \hat{X}^{(c)R} = R_{ij}\hat{X}_j^{(c)} \longrightarrow$ Transforms covariantly

Continued

$SO(3)$ TRANSFORMATION PROPERTIES OF H, SCHRODINGER ACTION

$$H = \frac{\hat{\vec{P}}^2}{2m} + V(\hat{X}_i); V(\hat{X}_i^R) = V(\hat{X}_i) \Rightarrow \hat{J}_i V(\hat{X}_i) = 0$$

$V(\hat{X}_i)/V(\hat{x}_i) \rightarrow$ operator in $\mathcal{H}_q^{(3)}/\mathcal{H}_c^{(3)}$

$$[\hat{J}_i, H] = [\hat{J}_i, \frac{\hat{\vec{P}}^2}{2m} + V(\hat{X}_i)] = [\hat{J}_i, V(\hat{x}_i)]$$

To compute $\hat{J}_i V(\hat{X}_i)\psi(\hat{x}_i) = \hat{J}_i(V(\hat{x}_i)\psi(\hat{x}_i))$

$$= (\hat{J}_i V(\hat{x}_i))\psi(\hat{x}_i) + V(\hat{x}_i)(\hat{J}_i\psi(\hat{x}_i)) +$$

$$\frac{1}{2}[(\hat{P}_i V)((\vec{\theta}.\hat{\vec{P}})\psi) - ((\vec{\theta}.\hat{\vec{P}})V)(\hat{P}_i\psi)]$$

Continued

Using $\hat{J}_i V = 0$

$[\hat{J}_i, H] = \frac{1}{2}[(\hat{P}_i V)((\vec{\theta}.\hat{\vec{P}})\psi) - ((\vec{\theta}.\hat{\vec{P}})V)\hat{P}_i]$

<u>ANOTHER PERSPECTIVE</u>

Note:$(V\psi)^R \neq V^R \psi^R = m[\Delta_0(R)(V \otimes \psi)]$

Rather, $(V\psi)^R = m[\Delta_\theta(R)(V \otimes \psi)] = V^R_{eff}\psi^R$

$= U(R)VU(R)^{-1}\psi^R$

$V^R_{eff} \rightarrow$ Effective potential in rotated frame.

Continued

SYMMETRY OF SCHRODINGER ACTION

$S = \int dt tr_c[\psi^\dagger(i\partial_t - \frac{\hat{\vec{P}}^2}{2m} - V(\hat{X}_i))\psi]$

is invariant under the following transf.

$\psi^\dagger \to (\psi^\dagger)^R = U(R)\psi^\dagger; \psi \to \psi^R = U(R)\psi, (V\psi) \to (V\psi)^R$

Since

$(\hat{J}_i\psi^\dagger)^\dagger = \frac{1}{2}\epsilon_{ijk}(\hat{x}_j(\hat{P}_k\psi^\dagger) + (\hat{P}_k\psi^\dagger)\hat{x}_j)^\dagger = -(\hat{J}_i\psi)$

$\Rightarrow (U(R)\psi^\dagger)^\dagger = (e^{i\vec{\phi}.\hat{\vec{J}}}\psi^\dagger)^\dagger = U(R)\psi$

$\Rightarrow tr_c((\psi^\dagger)^R\phi^R) = (U(R)\psi, U(R)\phi) = (\psi,\phi) = tr_c(\psi^\dagger\phi); U(R)^\ddagger U(R) = 1$

Continued

Although $\Delta_0(R)$ was used here implicitly, the same holds even if $\Delta_\theta(R)$ is used, as the additional terms are total commutators:
To summarise: S will be $SO(3)$ invariant, provided V also undergoes the transf.
$V \rightarrow V^R_{eff} = U(R)VU(R)^{-1}$

Even for $V^R(\hat{X}_i) = V(R^{-1}\hat{\vec{X}}_i)$. Generically, V^R_{eff} will have a reduced symmetry and therefore S will not be invariant.

Continued

EXAMPLE

$H = \frac{1}{2m}\hat{\vec{P}}^2 + \frac{1}{2}m\omega^2\hat{\vec{X}}^2 = \frac{1}{2m}\hat{\bar{\vec{P}}}^2 + \frac{1}{2}m\omega^2\hat{\bar{\vec{X}}}^2$
$= H_{plane} + H_{line}$
$[\hat{\bar{X}}_1, \hat{\bar{X}}_2] = i\theta; [\hat{\bar{X}}_1, \hat{\bar{X}}_3] = [\hat{\bar{X}}_2, \hat{\bar{X}}_3] = 0$
$\Psi_0(\hat{\bar{X}}_i) = e^{\frac{\alpha}{2\theta}(\hat{\bar{X}}_1^2+\hat{\bar{X}}_2^2)}e^{-\frac{1}{2}m\omega^2\hat{\bar{X}}_3^2} \to$ Has only $SO(2)$ symm.
To see it more explicitly, write
$\Psi_0 = \phi\psi; \phi = e^{\frac{\alpha}{2\theta}\hat{x}_i\hat{x}_i}, \psi = e^{\frac{\lambda}{2}x_3^2}$
Then $\hat{\bar{P}}\psi = \hat{\bar{J}}_i\phi = \hat{\bar{J}}_3\psi = 0$
Also, $\Delta_{\bar{\Theta}}(\hat{\bar{J}}_3) = \Delta_0(\hat{\bar{J}}_3) = \hat{\bar{J}}_3 \otimes 1 + 1 \otimes \hat{\bar{J}}_3$

Continued

$$\Rightarrow \hat{\bar{\bar{J}}}_3\Psi_0 = m[\Delta_{\bar{\Theta}}(\hat{\bar{\bar{J}}}_3)(\phi \otimes \psi)] =$$

$$m[\Delta_0(\hat{\bar{\bar{J}}}_3)(\phi \otimes \psi)] = 0$$

But $\hat{\bar{\bar{J}}}_\alpha\Psi_0 \neq 0$, As $\alpha = 1, 2$

Finally, this manifested in the explicit form

$$V^R_{eff}(\hat{X}_i) =$$

$$V(\hat{x}_i) + \frac{1}{2}m\omega^2[\frac{1}{4}((\vec{\theta}.\hat{\vec{\vec{P}}})^2 - (\vec{\theta}.\hat{\vec{\vec{P}}}^R)^2) - \theta_i(R_{ij} - \delta_{ij})\hat{J}_j]$$

Conclusions

We have discussed the generalization of non commutative quantum mechanics to three spatial dimensions.Particular attention was paid to the identification of the quantum Hilbert space and the representation of the rotation group on it. Not unexpectedly it was found that this representation undergoes deformation and that the angular momentum operators no longer obey the Leibnitz rule.

Continued

This deformation implies that the action for the Schroedinger equation, in which the potential appears as a fixed background field, and Hamiltonian are no longer invariant under rotations, even for rotational invariant potentials. This is in sharp contrast with the commutative case where rotational symmetry is manifest for rotational invariant potentials.