Two simple cases of interacting fermi gases

FTRTA2011
IISER-Kolkata
25th August, 2011

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Based on:

- (1) '*Gapped solitons and periodic excitations in strongly coupled BECs*', U. Roy, B. Shah, K. Abhinav and P. K. Panigrahi, J. Phys. B: At. Mol. Opt. Phys. **44** (2011) 035302.

- (2) '*Unitary Fermi Gas: Scaling Symmetries and Exact Map*', B. Chandrasekhar, K. Abhinav, V. M. Vyas and P. K. Panigrahi [Submitted to Euro. Phys. Lett.].

Outline of the talk

- BEC-BCS cross-over: The Unitarity
- BECs with solitonic solutions.
- Unitary Fermi gas: Conformal Non-relativistic symmetries
- Strongly coupled BEC with velocity restricted solutions
- Unitary fermi gas: Scaling symmetry
- Concluding remarks

References

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BEC-BCS cross-over: The Unitarity

- When cooled sufficienly, strongly interacting fermions become superfluid (Experimentally).
- The exact form of the interaction, and hence that of the system, depends crucially on the scattering length ‘a’.
- For negative a, corresponding attraction results into composite bosons: Cooper pairs.
- For positive a, repulsion allows loosely bound molecular states in vacuum: BEC.
- Through Feshbach resonance, ‘a’ can be smoothly varied.
- At the singular point, quasi-bound states appear: Unitarity.

References

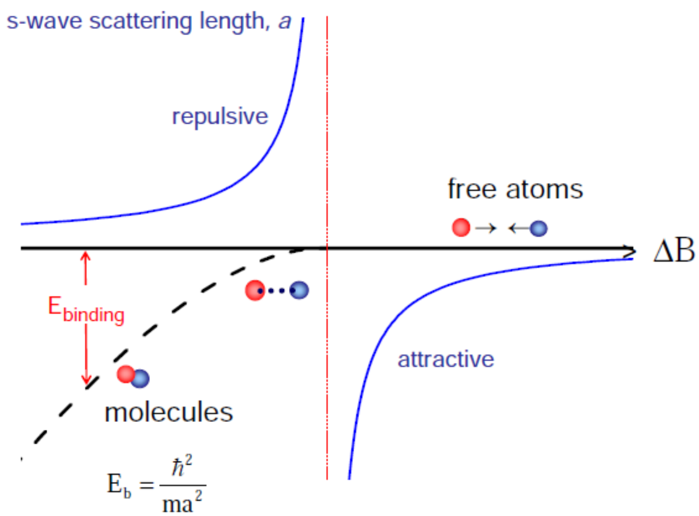

Fig.1: Controlling scattering length through Feshbach resonance.

- Resonance occurs when 'open' and 'closed' channel energies are close.
- Dilute : Interatomic potential range is far less than interparticle distance.
- Strongly interacting : Scattering length far greater than interparticle distance.

BECs with solitonic solutions

- Systems with four-Fermi self interaction are well-captured by mean-field approach at low energies.
- Macroscopic nature of the BECs allow the Gross-Pitaevskii equation to be applicable.
- When non-linearity balances the dispersion: Solitons.
- Solitons are familiar solutions of non-linear equations of varying orders.
- Close analogy with with ‘classical solutions’ of ‘phi-four’ field theory.

Unitary Fermi gas: Conformal Non-relativistic symmetries

- At Unitarity: Low energy admits non-relativistic behavior.
- Conformal symmetry prevails: Schroedinger algebra.
- A Heisenberg sub-algebra maps to oscillators with '2-Omega' modes in scaling parameter [1].
- Earlier observed as SU(1,1) '2-Omega' modes by Pitaevskii *et al.*.
- Scale-invariance: Universal nature of interaction: Effimov states and dimer formation.

Strongly coupled BEC with velocity restricted solutions

- The mean field behavior:
  - The N-particle Lagrangian with two-body interaction:

$$\frac{\hbar^2}{2m}\sum_{\sigma=1}^{N}\nabla\Psi_\sigma^\dagger.\nabla\Psi_\sigma+\sum_{\sigma=1}^{N}V_\sigma\Psi_\sigma^\dagger\Psi_\sigma+\sum_{\alpha,\beta}^{N}\Psi_\alpha^\dagger\Psi_\beta^\dagger U_{\alpha,\beta}\Psi_\beta\Psi_\alpha. \tag{1}$$

  - For a very large N, the equation of motion (GP Equation) is:

$$i\hbar\frac{\partial}{\partial t}\Psi_0=\left(-\frac{\hbar^2}{2m}\nabla^2+V+U|\Psi_0|^2\right)\Psi_0, \tag{2}$$

    depicting the Born-approximated mean field ground state behavior.

Strongly coupled BEC with velocity restricted solutions

- Cigar-shapped BEC:
  - The 'effective' external potential is:
$$V = \frac{1}{2} M \omega_\perp^2 (x^2 + y^2). \tag{3}$$
  - This allows a wave-function of the type:
$$\Psi(\mathbf{r}, t) = f(z, t) G(x, y, \sigma), \tag{4}$$
    where [2],
$$G(x, y; \sigma) = \frac{e^{-(x^2+y^2)/2\sigma^2}}{\pi^{1/2}\sigma},$$
$$\sigma(z) = \int dxdy |\Psi(x, y, z)|^2 = |f(z, t)|^2.$$
  - This yields the non-polynomial equation,
$$i\hbar \frac{\partial}{\partial t} f = \left[-\frac{\hbar^2}{2M} \frac{\partial^2}{\partial z^2} + \frac{U_0}{2\pi a_\perp^2} \frac{|f|^2}{\sqrt{1 + 2aN|f|^2}} + \frac{\hbar \omega_\perp}{2} \left(\frac{1}{\sqrt{1 + 2aN|f|^2}} + \sqrt{1 + 2aN|f|^2} \right) \right] f. \tag{5}$$

Strongly coupled BEC with velocity restricted solutions

- Strong and weak coupling limits:
  - In the strong-coupling limit,

$$2aN|f|^2 \gg 1, \quad N|\psi|^2 a \ll 1, \tag{6}$$

    to satisfy the diluteness of BEC.
  - This reduces the non-polynomial equation to:

$$i\hbar\frac{\partial}{\partial t}f = \left[-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial z^2} + 2\hbar\omega_\perp a^{1/2}\left(|f| - {\sigma_0}^{1/2}\right)\right] f. \tag{7}$$

  - In the weak-coupling limit:

$$2aN|f|^2 \ll 1, \tag{8}$$

    yielding,

$$i\hbar\frac{\partial}{\partial t}f = \left[-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial z^2} + 2\hbar\omega_\perp a\left(|f|^2 - \sigma_0\right)\right] f. \tag{9}$$

    which is the NLSE.

Strongly coupled BEC with velocity restricted solutions

- The ansatz and consistency:
  - The proposed ansatz is,

$$f(z,t) = e^{i(kz-\omega t)}\rho(\xi). \tag{10}$$

  - This leads to:

$$\alpha^2\rho'' + g\rho^2 + \epsilon\rho = 0, \tag{11}$$

    where,

$$\begin{aligned} g &= -4M\omega_\perp a^{1/2}/\hbar, \\ \epsilon &= 2M\omega/\hbar + 4M\omega_\perp(\sigma_0 a)^{1/2}/\hbar - k^2. \end{aligned}$$

  - Next ansatz:

$$\rho(\xi) = A + Bcn^2(\xi, m), \tag{12}$$

Strongly coupled BEC with velocity restricted solutions

- The ansatz and consistency (contd.):
  - The consistency conditions yield (1),

$$\begin{aligned} A &= \frac{1}{2g}\left[4\alpha^2(1-2m)-\epsilon\right], \quad B = \frac{6}{g}\alpha^2 m, \\ \epsilon^2 &= 16\alpha^4\left(m^2-m+1\right). \end{aligned}$$

- Stability and existance conditions:
  - For m=1, for the positive root of the effective chemical potential, we obtain a W-soliton as:

$$\rho(\xi) = -\frac{\epsilon}{g}\left[1-\frac{3}{2}sech^2(\xi)\right], \tag{13}$$

  - The corresponding Vakhitov-Kolokolov criterion reads:

$$\frac{dN(\epsilon)}{d\epsilon} = -\frac{6\epsilon}{g^2}. \tag{14}$$

    Thus, the solution is stable.
  - The restriction condition is:

$$k^2 \geq 2\frac{M\omega}{\hbar} - |\epsilon|. \tag{15}$$

Strongly coupled BEC with velocity restricted solutions

- Stability and existance conditions:

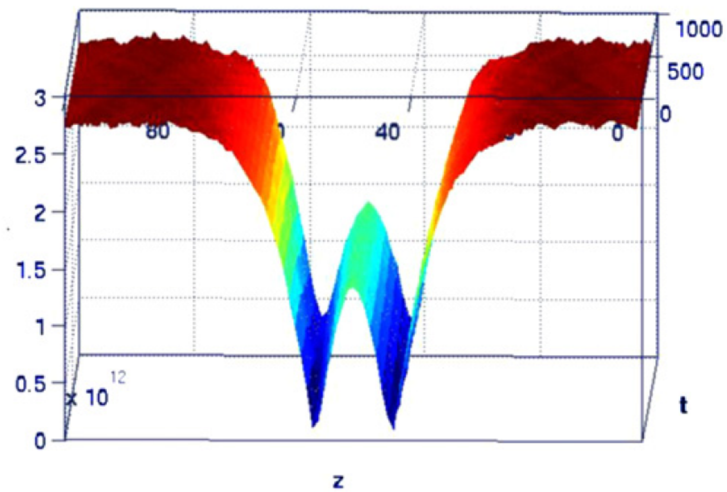

Fig.2:Numerical evolution of W-soliton depicting temporal stability.

  – For the negative root (m=1) one obtains,

$$\rho(\xi) = -\frac{3\epsilon}{2g} sech^2(\xi), \quad \frac{dN(\epsilon)}{d\epsilon} = 6\frac{\epsilon}{g^2},$$

$$k^2 \geq |\epsilon| + 2\frac{M\omega}{\hbar},$$

depicting a velocity restricted soliton for positive frequency.

Strongly coupled BEC with velocity restricted solutions

- The Pade'-type ansatz:
  - A more generic Pade'-type ansatz:

$$\rho(\xi) = \frac{A + B\ f(\xi)}{1 + C\ f(\xi)}, \tag{16}$$

yields localized solutions (m=1):

$$\rho(\xi) = -\frac{\epsilon}{g}\left(\frac{1 - 2sech(2\xi)}{1 + sech(2\xi)}\right), \tag{17}$$

which is the W-soliton obtained earlier.

  - Separatrix in the phase-space of the solutions.

Strongly coupled BEC with velocity restricted solutions

- Coherent control:
  - The re-casted GP-equation for strong coupling:

$$i\partial_t\psi = -\frac{1}{2}\partial^2_{zz}\psi + \gamma(t)|\psi|\psi + \frac{1}{2}M(t)z^2\psi + \frac{i\kappa(t)}{2}\psi. \tag{18}$$

  - Ansatz:

$$\psi(z,t) = B(t)F(\xi)e^{\left[i\Phi(z,t)+\frac{1}{2}G(t)\right]}, \quad \Phi(z,t) = a(t) + b(t)z - \frac{1}{2}c(t)z^2. \tag{19}$$

  - This yields a Riccati equation, which can be re-casted into a Scroedinger-like equation:

$$\begin{aligned} \frac{dc(t)}{dt} - c^2(t) &= M(t), \\ -\phi''(t) - M(t)\phi(t) &= 0, \quad c(t) = -\frac{\partial ln\phi(t)}{\partial t}. \end{aligned}$$

Strongly coupled BEC with velocity restricted solutions

- Coherent control (contd.):
  - Finally, we obtain the solutions as:

$$\psi(z,t) = -\frac{\epsilon}{g}\sqrt{A_0 sec(M_0 t)}[1-\frac{3}{2}sech^2(T/2)]e^{i\Phi(z,t)+\frac{1}{2}G(t)}, \quad M(t) = {M_0}^2,$$
$$\psi(z,t) = -\frac{\epsilon}{g}\sqrt{A_0 sech(M_0 t)}[1-\frac{3}{2}sech^2(T/2)]e^{i\Phi(z,t)}, \quad M(t) = -M_0^2.$$

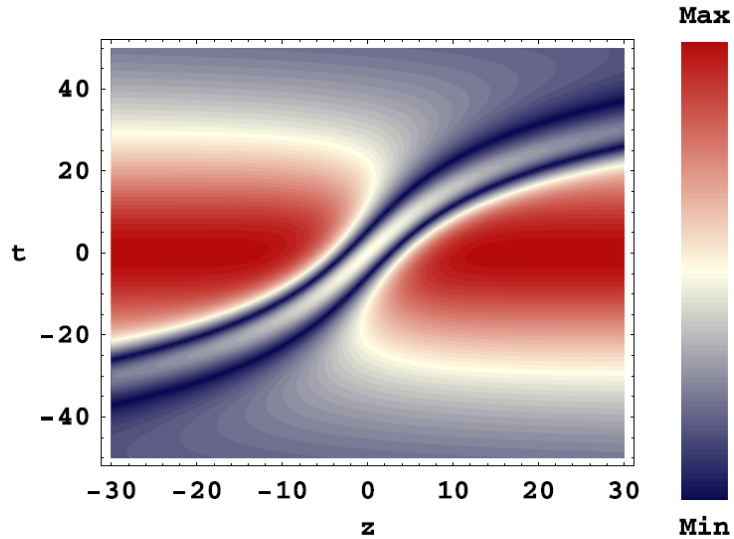

Fig.3: Temporal behavior of W-type soliton without gain/losss.

Unitary fermi gas: Scaling symmetry

- Fermions at unitarity:
  - Unitary fermi systems show scaling symmetry, enabling us to map it to free harmonic oscillators.

$$\sum_{i=1}^{N} \vec{r}_i \cdot \vec{\nabla}_i \psi = \gamma \psi. \tag{20}$$

  - The Universality of the system allows inverse-square two-body interactions, also supported by the existance of Efimov states [3].

$$\mathbf{r} \to \lambda \mathbf{r}, \quad \Psi(\mathbf{r}) \to \lambda^{d/2} \Psi(\lambda \mathbf{r}),$$
$$H \to \frac{1}{\lambda^2} \left(\sum_{i=1}^{N} \frac{P_i^2}{2m} + \sum_{i<j} V(\vec{r}_i - \vec{r}_j) \right)$$

  - In 2-D, Dirac delta interaction also have the same scaling property.
  - In 4-D, inverse-square wave-functions yield logarithmic divergences.

Unitary fermi gas: Scaling symmetry

- The Jastrow-type solution:
  - A harmonic trap:

$$H_{\text{trap}} = \sum_i \frac{1}{2} m\omega^2 \vec{r}_i^{\,2}, \tag{21}$$

    manifests a SO(2,1)/SU(1,1) symmetry, including the scaling operator.
  - Observed in 1-D Calogero-Sutherland model and in higher dimensions also.
  - In hyperspherical coordinates, the wave function admits a non-singular Jastrow factor:

$$\psi \equiv \prod_{i<j} |\vec{r}_i - \vec{r}_j|^{\beta}, \tag{22}$$

    and a Gaussian factor 'G' representing the trap dynammics.
  - The Jastrow factor generates a similarity transformation:

$$\begin{aligned} \tilde{H} &= \psi^{-1}\,(H)\,\psi = -\hat{A} + \epsilon_0, \\ \hat{A} &= \frac{1}{2}\sum_{i=1}^{N} \nabla_i^2 + \sum_{i=1}^{N} \vec{\nabla}_i(\ln\psi).\vec{\nabla}_i. \end{aligned}$$

Unitary fermi gas: Scaling symmetry

- The SU(1,1) algebra:
  - An interparticle potetial of the form:

$$V(\vec{r}_i - \vec{r}_j) = g^2 |\vec{r}_i - \vec{r}_j|^{-2}, \tag{23}$$

    leads to,

$$\tilde{H} = -\frac{1}{2}\sum_i \nabla_i^2 - \alpha \sum_{i\neq j} \frac{(\vec{r}_i - \vec{r}_j)}{|\vec{r}_i - \vec{r}_j|^2}\cdot\vec{\nabla}_i - \epsilon_0/2, \quad \alpha = (1+\sqrt{1+4g^2})/2. \tag{24}$$

  - There are two different ways to form the SU (1,1) algebra. The common two Cartan basis generators are:

$$T_0 = -\frac{1}{2}\left(\sum_i \vec{r}_i\cdot\vec{\nabla}_i + \epsilon_0\right), \quad T_- \equiv \frac{1}{2}\sum_i \vec{r}_i^2 \tag{25}$$

    and the choices of the third generator are:

$$T_+^f = \frac{1}{2}\sum_{i=1}^{N} \nabla_i^2, \quad T_+^i = \frac{1}{2}\sum_i \nabla_i^2 + \alpha \sum_{\substack{i,j=1\\ i\neq j}} \frac{(\vec{r}_i - \vec{r}_j)}{|r_i - r_j|^2}\cdot\vec{\nabla}_i. \tag{26}$$

Unitary fermi gas: Scaling symmetry

- The SU(1,1) algebra (Contd.):
  - Thus we obtain the SU(1,1) algebra:

$$[T_+, T_-] = -2T_0, \qquad [T_0, T_\pm] = \pm T_\pm. \tag{27}$$

  - Two algebras: The system is SI with or without the inter-particle interaction.
  - The algebra ensures the existace of an ‘omega’ breething mode.
  - The first SU(1,1) transformation yields:

$$e^{-T_-}\tilde{H}e^{T_-} \equiv \tilde{\tilde{H}} = \sum_{i=1}^{N} \vec{r}_i \cdot \vec{\nabla}_i - \hat{A}, \quad \epsilon_0 = \frac{1}{2}N + \frac{1}{2}N(N-1)\alpha, \tag{28}$$

  with a Universal scaling shift to the ground state energy.

Unitary fermi gas: Scaling symmetry

- The SU(1,1) algebra (Contd.):
  - Now, as:

$$[\tilde{\tilde{H}}, \exp\{-\hat{A}/2\}] = \hat{A}\exp\{-\hat{A}/2\}, \tag{29}$$

  - The operator:

$$\hat{T} \equiv \Psi_0 \exp\{-\hat{A}/2\}, \tag{30}$$

    diagonalizes the last Hamiltonian to:

$$H_D = \sum_i \vec{r}_i \cdot \vec{\nabla}_i + \epsilon_0, \tag{31}$$

    with polynomial eigenfunctions and ground energy shift of N/2 by the scaling exponent.
  - Successive transformations by 'free' raising operator and lowering operator yields:

$$H_{\text{decoupled}} = -\frac{1}{2}\sum_i \nabla_i^2 + \frac{1}{2}\sum_i \vec{r}_i^{\,2} + (\epsilon_0 - \frac{1}{2}N). \tag{32}$$

    with a shift to the ground state energy.

Unitary fermi gas: Scaling symmetry

- The origin of the omega mode:
  - In 1-D, the Jastrow-type symmetric polynomials can be formed without considering hyperspherical coordinaes.

$$\prod_l^N (x_i)^{n_l}, \quad \sum_l n_l\omega, \quad n_l = 0, 1, 2, \cdot \tag{33}$$

  - In higher dimensions, hyperspherical coordinates are required for symmetric polynomials.

$$\prod_l^N (r_i^2)^{n_l}, \quad E = 2\sum_l n_l + E_0. \tag{34}$$

    resulting into the '2 omega' modes.

Unitary fermi gas: Scaling symmetry

- Dimerization v/s molecule formation:
  - The equivalent three-body wave-function with the 'contact condition':

$$\psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \left(\frac{1}{r_{ij}} - \frac{1}{a}\right) A(\mathbf{R}_{ij}, \mathbf{r}_k) + O(r_{ij}). \tag{35}$$

  - Molecule formation needs non-zero 'A': Wave-function is singular at zero interparticle distance.
  - Dimerization requires A=0: Non-singular, symmetric Jastrow type wave function.
  - The scaling symmetry and the exact map is valid in the dimerization regime with long-distance correlations.
  - The Jastrow-type wave functions: Fractional exclusion statistics, in accord with the Monte Carlo simulations.

Concluding remarks

- Strong coupling BECs admit stable classical solutions of different forms. This considerably differs from the weak-coupling expectations. A complete quantum treatment can shed light, instead of a mean-field approach.

- New modes observed indicates lower dimensional uniqueness of unitary fermions. Non-perturbative treatment is on the cards.

- Mapping to simpler systems can mean an effective theory with observable quasi particles.

- 2+1 and 1+1 field-theoretic behavior is still to be studied, in face of present experimental realizations.

Thank you for your patience