Some Aspects of Chern-Simons Quantization

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Introduction

Dynamics in gauge theory is governed by the Yang-Mills Lagrangian density by which we mean the action is given by

$$
\begin{aligned}
S &= \int_M F \wedge {}^* F \\
&= \int_M \sqrt{-g} d^3x \ \frac{1}{2} tr F^{\mu\nu} F_{\mu\nu} \qquad (1)
\end{aligned}
$$

Action is metric dependent.
$tr T^a T^b = -\frac{1}{2}\delta^{ab}$, $A = A_\mu dx^\mu$
We apply the standard variational principle and apply usual procedure of Noether's theorem to find out energy momentum tensor.
In a topological field theory the action does not depend on the particular form of the space-time, it embraces all possible manifolds in a particular class , the energy momentum tensor is identically zero ; so to say $\frac{\delta S}{\delta g^{\mu\nu}} = 0$.

Chern-Simons action : $\Omega[A] = -\frac{1}{8\pi^2}\int_M (A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$
A topological gauge theory can be taken as,
$L = \frac{1}{2}\ trF^{\mu\nu}F_{\mu\nu} - \kappa\ tr(A_\alpha \partial_\beta A_\gamma + \frac{2}{3} A_\alpha A_\beta A_\gamma)$
This will describe a dynamical gauge theory affected by the topological term . If we consider a pure topological gauge theory there will be no dynamics, Hamiltonian is identically zero .
The above lagrangian is gauge invariant under small gauge transformation and the amplitude density $exp(iS)$ is invariant under all gauge transformation (small and large).
For pure Chern-Simons theory no dynamics – could one construct a general Hilbert space ? Here we need Geometric Quantization.

2+1 Gravity on a Torus

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Lagrangian with B-I like parameter

$$
\begin{aligned}
L &= L_1 + L_2 \\
L_1 &= \frac{1}{8\pi G} e^I \wedge \left(2 d\omega_I + \epsilon_I{}^{JK} \omega_J \wedge \omega_K + \frac{1}{3l^2} \epsilon_I{}^{JK} e_J \wedge e_K \right) \\
L_2 &= \frac{1}{8\pi G\gamma} (\omega^I \wedge d\omega_I + \frac{1}{l^2} e^I \wedge de_I + \frac{1}{3} \epsilon_{IJK} \omega^I \wedge \omega^J \wedge \omega^K \\
&+ \frac{1}{l^2} \epsilon_{IJK} \omega^I \wedge e^J \wedge e^K)
\end{aligned}
$$

γ is dimensionless. Internal metric = diag(-,+,+)

Chern Simons

- Choose SO(2,1) connection $A^I := \omega^I \pm \frac{e^I}{l}$
  then $L = \frac{l}{16\pi G}\left[\left(1/\gamma + 1\right) L + \left(1/\gamma - 1\right) L\right]$

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- Two $SO(2,1)$ Chern Simons Lagrangians.
- EOM: $\mathcal{F}_I := dA_I + \epsilon_{IJK} A^J \wedge A^K = 0$

$$\begin{aligned} 2d\omega_I + \epsilon_I{}^{JK}\omega_J \wedge \omega_K &= -\frac{1}{l^2}\epsilon_I{}^{JK} e_J \wedge e_K \\ de_I + \epsilon_{IJK} e^J \wedge \omega^K &= 0 \end{aligned}$$

Phase Space Structures

- Presymplectic structure on covariant phase space: $\Omega = \Omega^{(+)} + \Omega^{(-)}$

$$\Omega^{(\pm)}(\delta_1, \delta_2) = \frac{l}{8\pi G}\left(1/\gamma \pm 1\right)\int_\Sigma \delta_{[1} A^{(\pm)I} \wedge \delta_{2]} A_I^{(\pm)}$$

ie, $\{A_i^{(\pm)I}(x,t), A_j^{(\pm)J}(y,t)\} = \frac{8\pi G/l}{1/\gamma \pm 1}\varepsilon_{ij}\eta^{IJ}\delta^2(x,y)$

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- In terms of frame-connection variables:

$$\begin{aligned}
\{\omega_i^I(x,t), e_j^J(y,t)\} &= 4\pi G \frac{\gamma^2}{\gamma^2 - 1}\varepsilon_{ij}\eta^{IJ}\delta^2(x,y) \\
\{\omega_i^I(x,t), \omega_j^J(y,t)\} &= -4\pi G \frac{\gamma/I}{\gamma^2 - 1}\varepsilon_{ij}\eta^{IJ}\delta^2(x,y) \\
\{e_i^I(x,t), e_j^J(y,t)\} &= -4\pi G \frac{\gamma I}{\gamma^2 - 1}\varepsilon_{ij}\eta^{IJ}\delta^2(x,y)
\end{aligned}$$

Physical Phase Space

- Consider space-time foliable as $\Sigma \times \mathbb{R}$
  Physical phase space : space of flat Chern Simons Connections modulo gauge transformations

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- Induced symplectic structure:
  $\omega = \mathbf{d}\Theta = \frac{ik_{(+)}\pi}{4\tau_2} \mathbf{d}z_{(+)} \wedge \mathbf{d}\bar{z}_{(+)} + \frac{ik_{(-)}\pi}{4\tau_2} \mathbf{d}z_{(-)} \wedge \mathbf{d}\bar{z}_{(-)}$

Geometric Quantization of $\tilde{T}$

- Polarization: choice of (holomorphic) sections of line bundle over $\tilde{T}$: $\nabla_{\partial_{\bar{z}}}\Psi(z,\bar{z}) = 0$ where $\nabla = \mathbf{d} - i\Theta$

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- Physical state space should carry representations of:
  i) Heisenberg Weyl Group $\times$ ii) Fundamental group of $\tilde{T}$
  $\hat{\rho}' = -\frac{2i}{k}\tau\partial_z + \pi z \quad \hat{\sigma}' = \frac{2i}{k}\partial_z \qquad aba^{-1}b^{-1} = \Delta$
  $a\Delta a^{-1}\Delta^{-1} = 1 \; b\Delta b^{-1}\Delta^{-1} = 1$

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  - Hilbert space spanned by p states each q component.

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- $l/G \in \mathbb{Q}^+ \quad \gamma \in \mathbb{Q}^+$

Studies on consistent 3D Quantum gravity on Lens Spaces.

This is mainly a study on non-perturbative quantization of 3D gravity with positive cosmological constant (de Sitter space is the prototype vacuum solution - asymptotically ds . Lens space is topologically a three sphere modulo a discrete group . Eucleadianization of de Sitter is a three sphere. We show that in the first order formulation of gravity one can consistently carry out the calculation of the exact partition function by suitably augmenting the conventional theory an addtional topological term. The recent calculations by other authors show a severe divergence (nonregularizable) which is overcome by the choce of our action . In fact the introduce paramter helps us to tame the divergence .

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4. ‘Consistent 3D Quantum gravity on Lens Spaces’– Rudranil Basu and Samir K. Paul, (in preparation).

- $$Z = \int DA exp(i \frac{k}{4\pi} \int tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)) \tag{2}$$
- $$S[e, \omega] = 2 \int e^I \wedge (2d\omega_I + \epsilon_{IJK} \omega^J \omega^K) + \frac{1}{3q^2} \epsilon_{IJK} e^I \wedge e^J \wedge e^K \tag{3}$$
- $\Lambda = \frac{1}{q^2}$, $A^{(\pm)} = \omega \pm e/q$,
- $$S = q(I[A^+] - I[A^-]), \tag{4}$$
- $I[A] = \int \left(A^I \wedge dA_I + \frac{1}{3} \epsilon_{IJK} A^I \wedge A^J \wedge A^K \right)$

$dA_I^{(\pm)} + \epsilon_{IJK} A^{(\pm)J} \wedge A^{(\pm)K} = 0$, or
$de^I + \epsilon^{IJK} e_J \wedge \omega_K = 0$ and
$2d\omega^I + \epsilon^{IJK} \omega_J \wedge \omega_K = -\frac{1}{q^2} \epsilon^{IJK} e_J \wedge e_K$.

$$\begin{aligned}
\tilde{S}[e,\omega] &= 2q \int (\omega^I \wedge d\omega^I + \frac{1}{q^2} e^I \wedge de_I + \frac{1}{3} \epsilon_{IJK} \omega^I \wedge \omega^J \wedge \omega^K \\
&+ \frac{1}{q^2} \epsilon_{IJK} \omega^I \wedge e^J \wedge e^K) \\
&= q(I[A^+] + I[A^-]). \qquad (5)
\end{aligned}$$

same equations of motion.
$\tilde{I}[A^+, A^-] = \frac{k_+}{2\pi} I[A^+] + \frac{k_-}{2\pi} I[A^-]$, $k_\pm = \frac{q(1/\gamma \pm 1)}{8G}$.

$$Z^{\rm tot} = -\frac{1}{2\sqrt{r_+ r_-}} \sum_{P=1}^{\infty} \frac{1}{P} \sum_{\substack{Q({\rm mod} P)\\(Q,P)=1}} \exp\left(6\pi i s(Q,P)/R_+\right) \exp\{\frac{\pi i}{P}\left(2a + (Q+Q^*)\right)$$

$$(a/\gamma + 2)\} \times \left[\mathrm{e}^{\frac{\pi i}{PR_+} + \frac{2\pi i}{P}(Q+1)} + \mathrm{e}^{-\frac{\pi i}{PR_+} + \frac{2\pi i}{P}(Q-1)} - \mathrm{e}^{\frac{\pi i}{PR_-} + \frac{4\pi i}{P}} - \mathrm{e}^{\frac{-\pi i}{PR_-}}\right].$$

$\frac{1}{R_\pm} = \frac{1}{r_+} \pm \frac{1}{r_-}$, $r_\pm = k_\pm + 2$

Conclusion

There may be space time structure where canonical quantization is possible and partition function approach is almost impossible (anti-de Sitter) and there are certain space tiem structure where canonical approach fails and partition function is exactly calculable (de Sitter).