Nonrelativistic Chern-Simons Theory and Fractional Quantum Hall Effect in Graphene

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Plan of Talk

- Introduction to Chern-Simons Theory
- Introduction to Composite Fermion Theory in Fractional Quantum Hall Effect
- Role of CS Theory to Describe Composite Fermions
- Fractional Quantum Hall Effect in Graphene

Introduction to nonrelativistic Chern-Simons Theory

A collection of N point particles moving nonrelativistically on a plane and interacting each other through the mediation of U(1) gauge field, with Chern-Simons kinetic action.

$$H = \sum_{n=1}^{N} \frac{m}{2}(-i\hbar\vec{\nabla}_n + \vec{a}_n)^2$$

$$\vec{a}_n = \frac{1}{2\pi\kappa} \sum_{m\neq n}^{N} \frac{\hat{z}\times(\vec{r}_n-\vec{r}_m)}{|\vec{r}_n-\vec{r}_m|^2}$$

In two dimension, Green's function of a Laplacian,

$$G(\vec{r}-\vec{r}') = \frac{1}{2\pi}\ln\left(|\vec{r}-\vec{r}'|\right)$$

$$\nabla^2 G(\vec{r}-\vec{r}') = \delta(\vec{r}-\vec{r}')$$

$$\Rightarrow \vec{a}_n = \frac{1}{\kappa}\sum_{m\neq n}^{N} \hat{z}\times\vec{\nabla}G(\vec{r}_n-\vec{r}_m)$$

$$\Rightarrow b_n = \vec{\nabla}\times\vec{a}_n = \frac{1}{\kappa}\sum_{m\neq n}^{N}\delta(\vec{r}_n-\vec{r}_m)$$

The CS vector potential seen by particle n describes point vortices located at all other particles.

If we exchange two particles, the wave function will pick up a phase due to A-B effect. There statistics depends on the flux of the vortices. So the particles with CS interaction may have any statictics, i.e., anyons including bosons and fermions.

$$\vec{a}_n = -\frac{1}{2\pi\kappa}\vec{\nabla}_n \sum_{m\neq n}^{N} \theta(\vec{r}_n - \vec{r}_m), \quad \theta(\vec{r}) = \mathrm{Arctan}\,(y/x)$$

Pure gauge!

N-particle Schrodinger Equation for the wave function $\psi(\vec{r}_1, \cdots, \vec{r}_N; t)$:

$$i\hbar\partial_t\psi = \sum_{n=1}^{N} \frac{\hbar^2}{2m}\left[\vec{\nabla}_n - i\frac{e}{\hbar c}\vec{a}_n\right]^2 \psi$$

For stationary states, $\psi = e^{-iEt/\hbar}\phi(\vec{r}_1, \cdots, \vec{r}_N)$

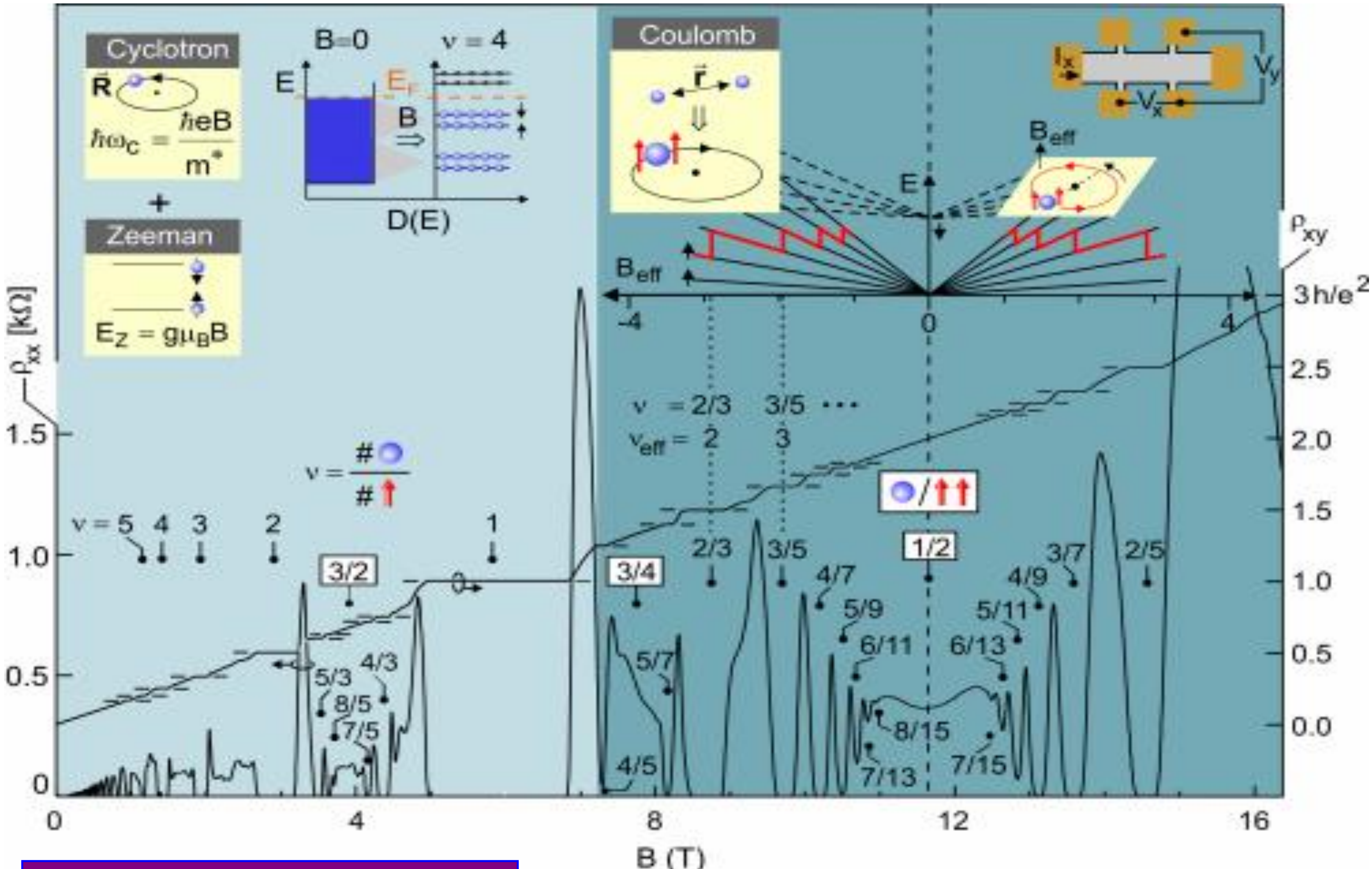
$$-\sum_{n=1}^{N} \frac{\hbar^2}{2m}\left[\vec{\nabla}_n - i\frac{e}{\hbar c}\vec{a}_n\right]^2 \phi = E\phi$$

$$\phi = \exp[-i\frac{e}{2\pi\hbar c\kappa}\sum_{m\neq n}\theta(\vec{r}_n - \vec{r}_m)]\phi_0$$

$$\Rightarrow -\sum_{n=1}^{N} \frac{\hbar^2}{2m}\nabla_n^2\phi_0 = E\phi_0$$

Free Schrodinger equation, but the information of interaction is hidden in the boundary condition satisfied by ϕ_0

Courtesy: J. SMET



Filling factor, $\nu = \frac{N}{N_\varphi} = \frac{\Phi_0 n_e}{B}$

$$\rho_{xy} = \frac{h}{fe^2}, \quad f = \frac{n}{2n+1}$$

Statement of the FQHE problem

• **Find the solutions for the quantum mechanical problem of interacting electrons in a magnetic field.**

$$H\Psi = E\Psi$$

$$H = \frac{1}{2m_b}\sum_j \left(\frac{\hbar}{i}\vec{\nabla}_j + \frac{e}{c}\vec{A}(\vec{r}_j)\right)^2 + \frac{e^2}{\epsilon}\sum_{j<k}\frac{1}{r_{jk}} + g\mu_B \vec{B}\cdot\vec{S}$$

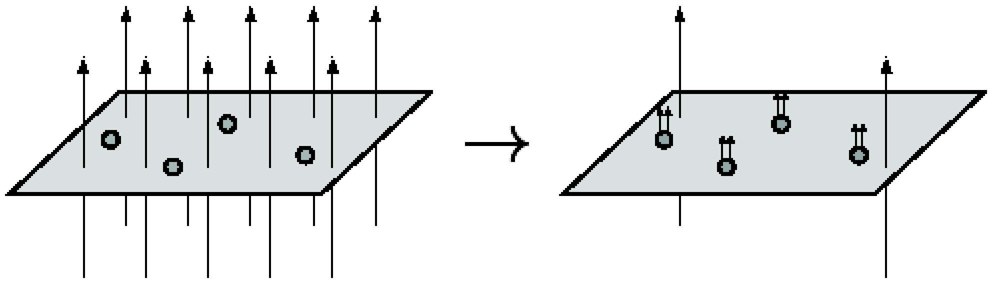
• **In the limit of $B \to \infty$ the kinetic and Zeeman energies are constant.**

$$H = \sum_{j<k}\frac{1}{r_{jk}} \qquad \text{(to be solved with the lowest LL restriction)}$$

What could be simpler?!!

The CF theory in a nutshell

Electrons transform into composite fermions by capturing 2p "flux quanta." Composite fermions experience a much reduced effective magnetic field. The complex problem of strongly correlated electrons thus maps into a simpler problem of weakly interacting fermions at an effective magnetic field.



$$B^* = B - 2p\rho\phi_0 \qquad \left(\phi_0 = \frac{hc}{e}\right)$$

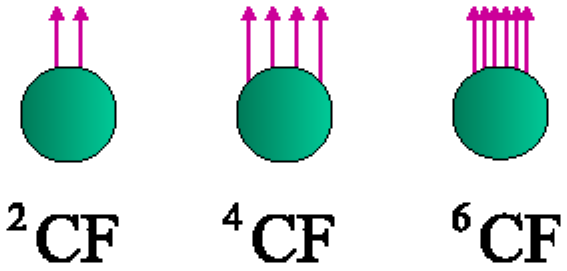
$$\nu = \frac{\nu^*}{2p\nu^* \pm 1}$$

$$\Psi_\nu = \mathcal{P}_{\rm LLL}\Phi_{\nu^*}\prod_{j<k}(z_j - z_k)^{2p}$$

The CF theory makes precise, falsifiable statements about both the microscopic solution and its physical meaning. Both must be verified.

J K Jain, 1989

Composite fermions



Composite fermion = electron + 2p quantized vortices

A quantized vortex is a topological object. Hence also is the composite fermion. The vorticity 2p is a topological "charge" of the composite fermion.

A vortex is often represented as a flux quantum. A composite fermion is often thought of (somewhat inaccurately) as:

Composite fermion = electron + 2p flux quanta

Courtesy: J K Jain

Chern Simons transformation

Jain; Lopez, Fradkin; Zhang, Hansson, Kivelson

$$H\Psi = E\Psi$$

$$H = \sum_j \frac{1}{2m_b}\left[\frac{\hbar}{i}\nabla_j + \frac{e}{c}\mathbf{A}(\mathbf{r}_j)\right]^2 + \frac{e^2}{\epsilon}\sum_{j<k}\frac{1}{|\mathbf{r}_j - \mathbf{r}_k|}$$

$$\Psi = \prod_{j<k}\left(\frac{z_j - z_k}{|z_j - z_k|}\right)^{2p}\Psi' = \exp^{-i\sum_{j<k}\phi_{jk}}\ \Psi' \qquad \phi_{jk} = i\ln\frac{z_j - z_k}{|z_j - z_k|}$$

$$H'\Psi' = E\Psi' \qquad z_j = x_j - iy_j$$

$$H' = \left[\frac{1}{2m_b}\sum_i\left(\vec{p}_i + \frac{e}{c}\vec{A}(\vec{r}_i) - \frac{e}{c}\vec{a}(\vec{r}_i)\right)^2 + V\right]$$

$$\vec{a}(\vec{r}_i) = \frac{2p}{2\pi}\phi_0 \sum_j{}' \vec{\nabla}_i\phi_{ij}$$

$$\vec{b}_i = \vec{\nabla}_i \times \vec{a}(\vec{r}_i) = 2p\phi_0 \sum_l{}' \delta^2(\vec{r}_i - \vec{r}_l)$$

The Chern-Simons transformation attaches 2p point "flux quanta" to each electron to convert it into a composite fermion.

Mean field approximation

So far, everything is exact. Now we make a mean-field approximation by spreading the point flux on each electron into a uniform magnetic field.

$$\vec{A} - \vec{a} = \vec{A} - \langle \vec{a} \rangle - \delta\vec{a} \equiv \vec{A}^* - \delta\vec{a}$$

$$\vec{\nabla} \times \vec{A}^* = B^* \hat{\vec{z}}$$

$$H' = \frac{1}{2m_b} \sum_i \left(\vec{p}_i + \frac{e}{c} \vec{A}^*(\vec{r}_i) \right)^2 + V + V' = H'_0 + V + V'$$

The first term has known solutions (non-interacting particles in an effective field). The other two are to be treated perturbatively (assuming that the perturbation theory converges). The "unperturbed" mean-field solution is given by:

$$\Psi'_{MF} = \Phi_n$$

$$\Psi_{MF} = \Phi_n \prod_{j<k} \left(\frac{z_j - z_k}{|z_j - z_k|} \right)^{2p}$$

Because of the absence of a small parameter, it is not possible to obtain quantitative information from the Chern-Simons approach. But it suggests good variational wave functions.

The "final" wave function $\left(\nu = \frac{n}{2pn+1}\right)$

$$\Psi_{MF} = \Phi_n \prod_{j<k} \left(\frac{z_j - z_k}{|z_j - z_k|} \right)^{2p}$$

No good. It does not have good correlations, and has much amplitude in higher Landau levels.

Try $$\Psi_\nu = \mathcal{P}_{\mathrm{LLL}} \Phi_n \prod_{j<k} (z_j - z_k)^{2p}$$

$$z_j = x_j - iy_j$$

- **The wave function appears simple, but represents very complicated correlations.**
- **Generalization to excited states is straightforward.**

In the mean CS magnetic field approximation, the CS theory does not provide good many body wave function that predicted in composite fermion theory.

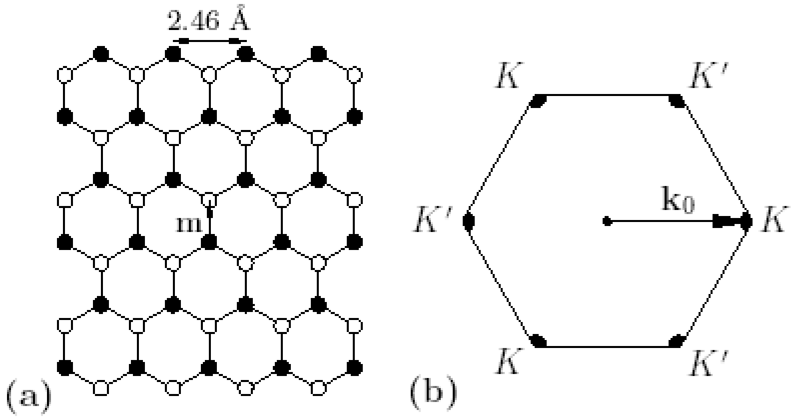
Considering Gaussian fluctuation of CS gauge field over the mean magnetic field, one can determine density distribution (square of the modulus of the wave function). This reproduces the missing correlation that is important in CF theory. However, the wave function becomes multivalued when n>1, because the exponent of the Jastrow form becomes 1/n.

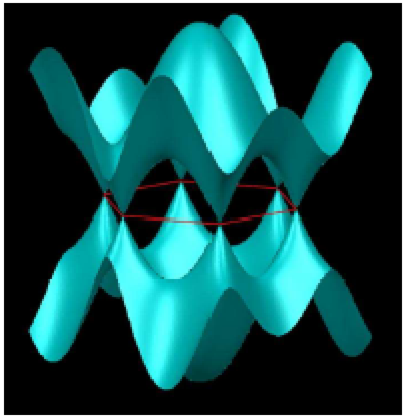
$$\Psi_{\mathrm{CS}}^{\frac{n}{2pn+1}} = \prod_{i<j}^{N}(z_i - z_j)^{2p+1/n}$$

$$\Psi_{\mathrm{CF}}^{\frac{n}{2np+1}} = \mathcal{P}\Phi_n \prod_{i<j}^{N}(z_i - z_j)^{2p}$$

For multicomponent fermions, CS coupling becomes a matrix. Using this matrix, one can determine possible filling factors and the corresponding wavefunctions (with the help of CF theory regarding integer filling part of the wave function) may be predicted.

Graphene dispersion: 2D massless Dirac fermions





Two sublattices: A and B Hamiltonian: $H = \begin{pmatrix} 0 & t_k \\ t_k^* & 0 \end{pmatrix}$

$t_k = t\left[1 + 2e^{i(\sqrt{3}/2)k_y a}\cos(k_x a/2)\right]$ Spectrum $\varepsilon_k^2 = |t_k|^2$

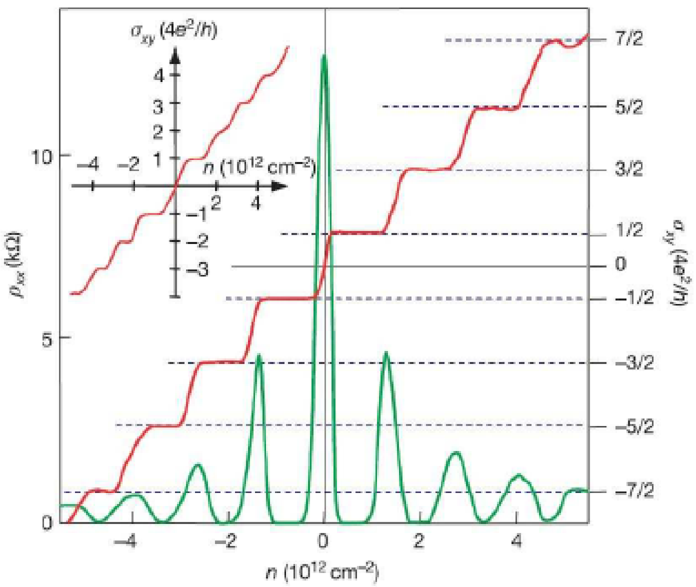
The gap vanishes at 2 points, $K, K' = (\pm k_0, 0)$, where $k_0 = 4\pi/3a$.

In the vicinity of K, K' the spectrum is of massless Dirac-fermion type:

$$H_K = v_0(k_x\sigma_x + k_y\sigma_y), \qquad H_{K'} = v_0(-k_x\sigma_x + k_y\sigma_y)$$

$v_0 \simeq 10^8$ cm/s – effective "light velocity", sublattice space $\longrightarrow$ isospin

Graphene in transverse magnetic field



Anomalous integer quantum Hall effect: $\sigma_{xy} = (n + \frac{1}{2}) \times (4e^2/h)$

$4 = 2$ (spin) $\times$ 2 (number of Dirac points)

Observed up to room temperature !

FQHE in 1/3, 2/3 etc filling factors have also been observed.

Fermionic CS theory in Graphene with SU(4) symmetry

Filling factor ν <1 in n=0 Landau level :

$$\mathcal{H} = \int d\vec{r}\psi_e^\dagger(\vec{r}) H \psi_e(\vec{r}) + \frac{1}{2}\int d\vec{r}\int d\vec{r}' V(\vec{r}-\vec{r}') : \hat{\rho}_e(\vec{r})\hat{\rho}_e(\vec{r}') :$$

$$\psi_e = \begin{pmatrix} \psi_e^{(+\uparrow)} \\ \psi_e^{(+\downarrow)} \\ \psi_e^{(-\uparrow)} \\ \psi_e^{(-\downarrow)} \end{pmatrix} \equiv \begin{pmatrix} \psi_e^{(1)} \\ \psi_e^{(2)} \\ \psi_e^{(3)} \\ \psi_e^{(4)} \end{pmatrix}$$

$$H = v_F(\sigma_x \Pi_x + \tau_z \sigma_y \Pi_y)$$

$$\vec{\Pi} = -i\vec{\nabla} + e\vec{A}, \quad \vec{\nabla}\times\vec{A} = B\hat{z}$$

$$\psi_{e,\alpha}^\dagger(\vec{r}) \rightarrow \psi_\alpha^\dagger(\vec{r}) e^{-i\mathcal{K}_{\alpha\beta}\int d\vec{r}' \arg(\vec{\mathrm{r}}-\vec{\mathrm{r}}')\rho_\beta(\vec{\mathrm{r}}')}$$

$$H_{\text{eff}} = \int d\vec{r}\,\psi_\alpha^\dagger(\vec{r})v_F\left(\sigma_x\tilde{\Pi}_{\alpha,x} + \tau_z\sigma_y\tilde{\Pi}_{\alpha,y}\right)\psi_\alpha(\vec{r})$$

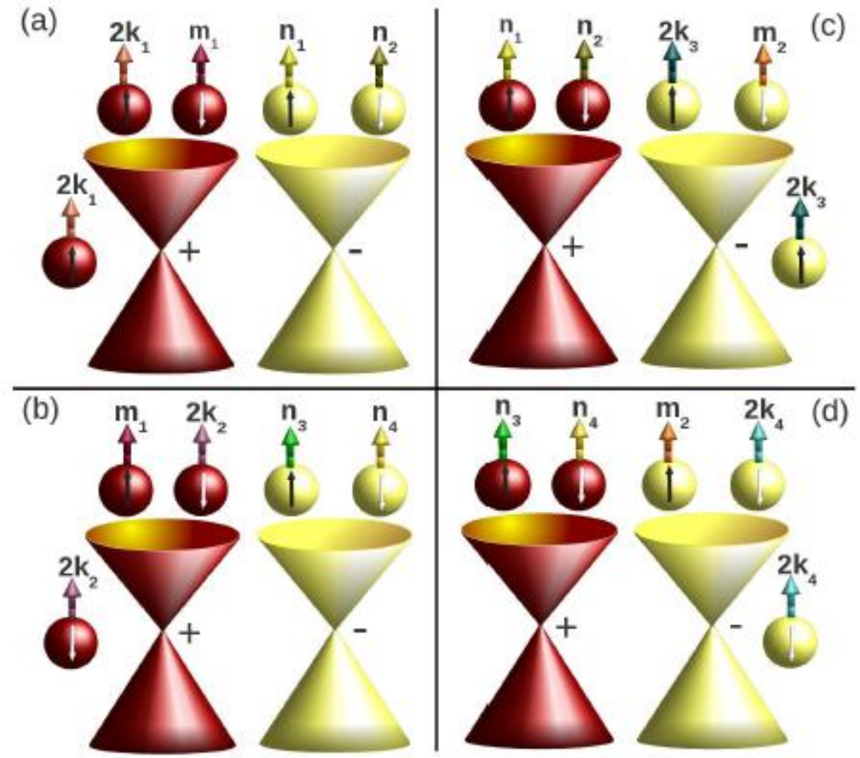
$$\vec{\tilde{\Pi}}_\alpha = -i\vec{\nabla}+e\vec{A}-\vec{a}_\alpha\ ,\ \ \vec{a}_\alpha = \mathcal{K}_{\alpha\beta}\int d\vec{r}'g(\vec{r}-\vec{r}')\rho_\beta(\vec{r}')$$

$$g(\vec{r}) = (\hat{z}\times\vec{r})/r^2$$

$$b_\alpha \equiv (1/e)\vec{\nabla}\times\vec{a}_\alpha = \phi_0\mathcal{K}_{\alpha\beta}\ \rho_\beta(\vec{r})$$

$$\mathcal{K} = \begin{pmatrix} 2k_1 & m_1 & n_1 & n_2 \\ m_1 & 2k_2 & n_3 & n_4 \\ n_1 & n_2 & 2k_3 & m_2 \\ n_3 & n_4 & m_2 & 2k_4 \end{pmatrix},$$

$$\rho_\alpha/\nu_\alpha = \rho/\nu - \mathcal{K}_{\alpha\beta}\rho_\beta$$



$$\rho_\alpha/\nu_\alpha = \rho/\nu - \mathcal{K}_{\alpha\beta}\rho_\beta$$

$$S = (\rho_1 + \rho_3 - \rho_2 - \rho_4)/\rho,$$
$$V = (\rho_1 + \rho_2 - \rho_3 - \rho_4)/\rho,$$
$$M = (\rho_1 + \rho_4 - \rho_2 - \rho_3)/\rho$$

$$\begin{aligned}
\gamma_1(2k_1 + \frac{1}{\nu_1}) + \gamma_2 m_1 + \eta_1(n_1 + n_2) + \delta_1(n_1 - n_2) &= \frac{4}{\nu}, \\
\gamma_2(2k_2 + \frac{1}{\nu_2}) + \gamma_1 m_1 + \eta_1(n_3 + n_4) + \delta_1(n_3 - n_4) &= \frac{4}{\nu}, \\
\gamma_3(2k_3 + \frac{1}{\nu_3}) + \gamma_4 m_2 + \eta_2(n_1 + n_2) + \delta_2(n_1 - n_2) &= \frac{4}{\nu}, \\
\gamma_4(2k_4 + \frac{1}{\nu_4}) + \gamma_3 m_2 + \eta_2(n_3 + n_4) + \delta_2(n_3 - n_4) &= \frac{4}{\nu},
\end{aligned}$$

$\eta_1 = 1 - V$, $\eta_2 = 1 + V$, $\delta_1 = S - M$, $\delta_2 = S + M$,
$\gamma_1 = \eta_2 + \delta_2$, $\gamma_2 = \eta_2 - \delta_2$, $\gamma_3 = \eta_1 + \delta_1$, and $\gamma_4 = \eta_1 - \delta_1$

$$\Psi_\nu(\{u^\alpha\}) = \mathcal{P}_L\Big[\prod_{\alpha=1..4}\Phi_{\nu_\alpha}(u_1^\alpha,...,u_{N_\alpha}^\alpha)\Big]\prod_{i<j}^{N_\alpha}(u_i^\alpha - u_j^\alpha)^{2k_\alpha}$$

$$\times \prod_{i,j;\alpha,\beta;\alpha\neq\beta}^{N_\alpha,N_\beta}(u_i^\alpha - u_j^\beta)^{\mathcal{K}_{\alpha\beta}}$$

$$u^\alpha = x^\alpha - iy^\alpha$$

$$n_1 = n_2 = n_3 = n_4 = n\,,\ k_1 = k_2,\ k_3 = k_4$$

$$\mathrm{Case-I:}\ \ 2k_1 = 2k_3 = m_1 = m_2 = n = 2k$$

$$\nu = \nu^*/(2k\nu^*+1),\ V = (2(\nu_1+\nu_2)-\nu^*)/\nu^*,$$
$$S = (2(\nu_1+\nu_3)-\nu^*)/\nu^*,\ M = (2(\nu_1+\nu_4)-\nu^*)/\nu^*$$

Conventional composite fermion states are recovered.

$$\mathrm{If}\ \nu_1 = \nu_2 = \nu_3 = \nu_4 = 1\,(\mathrm{SU(4)Singlet}),$$
$$M = S = V = 0\ \mathrm{and}\ \nu = \tfrac{4}{8k+1}$$

$\mathrm{Case-II}: \nu_1 = \nu_2 = \nu_3 = \nu_4 = 1$

(A) $M = S = V = 0$

$$\nu = \frac{8}{(2+2k_1+2k_3+m_1+m_2+4n)},$$
$$\text{when}\, 2k_1 + 1 \neq m_1,\ \text{and}\, 2k_3 + 1 \neq m_2$$

Unconventional state like 4/11 appears!

(B) $M = S = 0,\ \text{but}\, V \neq 0$

Unconventional states like 1/2, 3/8, 3/5, 3/7, 4/7 appear

(C) M and S undetermined

$$k_1 = k_3 = k\, \text{and}\, 2k + 1 = m_1 = m_2 = m,\ \nu = \frac{2}{m+n}$$

V is also undetermined when m=n: 1/3, 1/5

V=0 otherwise: 1/2, 2/3, 2/5, 1/3, 1/4

Conclusion

All the states are obtained for IQHE of composite fermions, although most of the states are not included in conventional composite fermion theory. The present formalism suggests that the composite fermion picture is more robust.

The validity of new states in the realistic two body interaction potential should be tested.

Experiments need to be performed to search for the new predicted states.