LOFF Phase in Noncommutative Field Theories

Hiranmaya Mishra, P. K. Panigrahi, and **T. Shreecharan**

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Whole of quantum mechanics rests on three canonical commutation relations

$$[\hat{x}^i, \hat{x}^j] = [\hat{p}^i, \hat{p}^j] = 0, \quad [\hat{x}^i, \hat{p}^j] = i\hbar\delta^{ij}.$$

The consequences of these commutation relations is well-known. We wish to remind that immediate consequence of this is the Heisenberg's uncertainty relation

$$\Delta x \Delta p \geq \hbar/2.$$

This means that one cannot localize the points of phase space as in the classical world.

History of noncommutative physics

- Historically the idea of a NC space-time was proposed by Heisenberg in a letter to Peierls.
- Peierls described this idea to Pauli who in turn told it to Oppenheimer.
- Oppenheimer then told about this, to his graduate student H. S. Snyder, who wrote the first full length article on this subject in 1947.
- C. N. Yang the Nobel laureate had also considered such a proposal at the same time.

The Original

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Quantized Space-Time

HARTLAND S. SNYDER
Department of Physics, Northwestern University, Evanston, Illinois

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It is usually assumed that space-time is a continuum. This assumption is not required by Lorentz invariance. In this paper we give an example of a Lorentz invariant discrete space-time.

THE problem of the interaction of matter and fields has not been satisfactorily solved to this date. The root of the trouble in present field theories seems to lie in the assumption of point interactions between matter and fields. On the other hand, no relativistically invariant Hamiltonian theory is known for any form of interaction other than point interactions.

Even for the case of point interactions the relativistic invariance is of a formal nature only, as the equations for quantized interacting fields have no solutions. The uses of source functions, or of a cut-off in momentum space to replace infinity by a finite number are distasteful arbitrary procedures, and neither process has yet been formulated in a relativistically invariant manner. It may not be possible to do this.

It is possible that the usual four-dimensional continuous space-time does not provide a suitable framework within which interacting matter and fields can be described. I have chosen the idea that a modification of the ordinary concept of space-time may be necessary because the "elementary" particles have fixed masses and associated Compton wave-lengths.

The special theory of relativity may be based on the invariance of the indefinite quadratic form

$$S^2 = c^2t^2 - x^2 - y^2 - z^2, \qquad (1)$$

Star product

The fact that we have replaced commuting coordinates with their NC counterparts (operators) is not very convenient for field theoretical calculations. It is convenient to work with functions, instead of operators, but with a new kind of product called the "Star" product.

$$
\begin{aligned}
f(\hat{x})g(\hat{x}) &= \exp\left[\frac{i}{2}\Theta^{ij}\partial_i^1\partial_j^2\right] f(x_1)g(x_2)\,|_{x_1=x_2=x} \\
&= f(x)\star g(x).
\end{aligned}
$$

Properties of star product

1. Associativity: $[(f \star g) \star h] = [f \star (g \star h)]$.
2. Cyclicity:
   $\int d^4x(f_1 \star f_2 \star \cdots \star f_n)(x) = \int d^4x(f_n \star f_1 \star \cdots \star f_{n-1})(x)$.
3. Complex conjugation: $(f \star g)^* = g^* \star f^*$.
4. Bilinears: $\int d^4x(f \star g) = \int d^4x(g \star f) = \int d^4x(fg)$.
5. Exponentials: $e^{ipx} \star e^{ikx} = e^{i(p+k)x} e^{ip^i \Theta_{ij} k^j/2}$.

Vacuum structure of NC field theory

- Well known phenomena of Superconductivity
- Superconductivity results due to formation of Cooper pairs.
- Cooper pairs characterized by fermion pairs with equal and opposite momentum.
- The order parameter is constant. $\langle\psi\psi\rangle = \Delta$.
- Δ is called the gap parameter. The minimum required energy to excite particles in the superconducting state.

The LOFF phase

- Quite an old result dating back to 1960's. Discovered Independently by Larkin-Ovchinnikov and Fulde-Ferrel while studying role of impurities in a superconductor.
- Fermion pairs characterized by non-zero net momentum.
- Density imbalance or in other words two fermionic species necessary for the formation of pairs.
- Order parameter is no longer a constant.

$$\Delta(r) = \Delta\, e^{iq\cdot r}.$$

- Possible occurrence in dense quark matter, cold fermionic gases and Heavy fermionic superconductors.

The Noncommutative Hamiltonian

The Hamiltonian is given by

$$\mathcal{H} = \psi_r^\dagger \left[-\frac{\nabla^2}{2m} - \mu \right] \psi_r - \frac{g}{2} \psi_r^\dagger \star \psi_r \star \psi_s^\dagger \star \psi_s$$

We need to find out the ground state.

$$\mid \Omega(\beta,\, \mu)\rangle = \mathcal{U}_{\beta,\,\mu} \mid \Omega\rangle_{\mathbf{q}} = \mathcal{U}_{\beta,\,\mu} \mathcal{U}_{\mathbf{q}} \mid 0\rangle.$$

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$$\begin{aligned}
\mathcal{U}_{\mathbf{q}} &= \exp\left[B_{\mathbf{q}}^\dagger - B_{\mathbf{q}} \right] \quad \mathcal{U}_{\beta,\,\mu} = \exp\left[B_{\beta,\,\mu}^\dagger - B_{\beta,\,\mu} \right] \\
B_{\mathbf{q}}^\dagger &= \int d^3k\, \psi_r^\dagger(\mathbf{k} + \frac{\mathbf{q}}{2})\, f(\mathbf{k})\, \psi_{-r}^\dagger(-\mathbf{k} + \frac{\mathbf{q}}{2}) \\
B_{\beta,\,\mu}^\dagger &= \int d\mathbf{k}\, \widetilde{\psi}_r^\dagger(\mathbf{k}) \theta(\mathbf{k}, \beta, \mu) \widetilde{\phi}_r^\dagger(\mathbf{k})
\end{aligned}$$

Bogoliubov Transformation

$$\left[\begin{array}{c} \widetilde{\psi}_r(\mathbf{k}) \\ \widetilde{\psi}^{\dagger}_{-r}(-\mathbf{k}+\mathbf{q}) \end{array}\right] =$$

$$\left(\begin{array}{cc} \cos f(\mathbf{k}-\mathbf{q}/2) & -2r\sin f(\mathbf{k}-\mathbf{q}/2) \\ 2r\sin f(\mathbf{k}-\mathbf{q}/2) & \cos f(\mathbf{k}-\mathbf{q}/2) \end{array}\right)\left[\begin{array}{c} \psi_r(\mathbf{k}) \\ \psi^{\dagger}_{-r}(-\mathbf{k}+\mathbf{q}) \end{array}\right].$$

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$$\begin{bmatrix} \psi_r(\mathbf{k},\beta) \\ \phi^\dagger_r(\mathbf{k},\beta) \end{bmatrix} = \begin{pmatrix} \cos\theta(\mathbf{k},\beta) & -\sin\theta(\mathbf{k},\beta) \\ \sin\theta(\mathbf{k},\beta) & \cos\theta(\mathbf{k},\beta) \end{pmatrix} \begin{bmatrix} \widetilde{\psi}_r(\mathbf{k}) \\ \widetilde{\phi}^\dagger_r(\mathbf{k}) \end{bmatrix}$$

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Expectation values

$$\Omega = \langle \mathcal{H}_0 \rangle + \langle \mathcal{H}_I \rangle - \frac{1}{\beta}\,\mathcal{S}.$$

$$\mathcal{S} = -\sum_{a=\pm} \int \frac{d\mathbf{P}}{(2\pi)^3} [\,\sin^2\theta_a \ln \sin^2\theta_a + \cos^2\theta_a \ln \cos^2\theta_a].$$

$$\langle \mathcal{H}_0 \rangle = \int \frac{d\mathbf{P}}{(2\pi)^3} [\epsilon(\mathbf{P}_+) - \mu] \left[\sin^2\theta_+ + \sin^2 f(\mathbf{P}) \left(1 - \sin^2\theta_- - \sin^2\theta_+\right)\right]$$
$$+ \int \frac{d\mathbf{P}}{(2\pi)^3} [\epsilon(\mathbf{P}_-) - \mu] \left[\sin^2\theta_- + \sin^2 f(\mathbf{P}) \left(1 - \sin^2\theta_- - \sin^2\theta_+\right)\right].$$

$$\langle \psi_r^\dagger(\mathbf{x}) \psi_s^\dagger(\mathbf{x}) \rangle = r\,\delta_{-r,s} e^{-i\,\mathbf{q}\cdot\mathbf{x}} \int \frac{d\mathbf{P}}{(2\pi)^3}\, \sin 2f(\mathbf{P}) \left[1 - \sin^2\theta_- - \sin^2\theta_+\right].$$

The gap equation

$$\tan 2f(\mathbf{k}) = \cos\left[\frac{\mathbf{k}\times\mathbf{q}}{2}\right]\frac{\Delta_0}{\xi(\mathbf{k},\mathbf{q})}.$$

$$\Delta_0 = -\frac{g}{2}\int\frac{d\mathbf{P}}{(2\pi)^3}\,\cos^2\left[\frac{\mathbf{P}\times\mathbf{q}}{2}\right]\frac{\Delta_0}{\omega_q}\left[1-\sin^2\theta_- - \sin^2\theta_+\right].$$

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$$\omega_q = \left\{\left[\frac{(P^2+q^2/4)}{2m}-\mu\right]^2 + \Delta_0^2\cos^2\left[\frac{\mathbf{P}\times\mathbf{q}}{2}\right]\right\}^{1/2}$$

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$$\sin^2\theta_\pm = \frac{1}{\exp(\beta\,\omega_\pm)+1}.$$

Solution of the gap equation

$$
\begin{aligned}
\mathcal{F}(\Theta) &= \cos^2\left[\frac{\mathbf{P}\times\mathbf{q}}{2}\right]\frac{1}{\omega_q}. \\
&\approx \mathcal{F}(0) + \Theta\,\mathcal{F}'(\Theta)|_{\Theta=0} + \frac{\Theta^2}{2}\mathcal{F}''(\Theta)|_{\Theta=0}.
\end{aligned}
$$

Solution of the gap equation

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$$\approx \mathcal{F}(0) + \Theta\,\mathcal{F}'(\Theta)|_{\Theta=0} + \frac{\Theta^2}{2}\mathcal{F}''(\Theta)|_{\Theta=0}.$$

$$1 = -\frac{g}{2}\int\frac{d\mathbf{P}}{(2\pi)^3}\left\{\frac{1}{\omega_0} + \frac{P^2q^2\Theta^2}{12}\left[-\frac{1}{\omega_0} + \frac{\Delta_0^2}{2\omega_0^3}\right]\right\}$$
$$[1 - H(-\omega_-) - H(-\omega_+)]\,.$$

Example

$$I = \int \frac{d\mathbf{P}}{(2\pi)^3} \frac{1}{\omega_0} H(-\omega_\pm).$$

Where $-\omega_\pm = \omega \pm \delta_\xi$. With $\delta_\xi = (\xi_+ - \xi_-)/2 \equiv (P\, q\, \cos\eta)/2m$. Let us introduce scaled variables in terms of the Fermi momentum as

$$P = x\, P_F, \quad q = y\, P_F, \quad \mu = \hat{\mu}\, \epsilon_F, \quad \Delta_0 = z\, \epsilon_F, \quad \epsilon_F = P_F^2/2m.$$

$$x_{max/min} = \sqrt{\nu} \pm \frac{1}{2\sqrt{\nu}} \sqrt{\delta_t^2 - z^2} \quad |\delta_t| > z.$$

$$I = \int_{x_{min}}^{x_{max}} dx\, dt\, \frac{x^2}{\left[(x^2-\nu)^2 + z^2\right]^{1/2}} H(|\delta_t| - z).$$

Solution contd..

In the weak coupling scenario the numerator $x^2 \approx \nu$. In this case the contribution to the integral is also maximum thus $\omega_0 \equiv \sqrt{4\nu(x-\sqrt{\nu})^2+z^2}$. The integral can be solved by changing the variable $2\sqrt{\nu}(x-\sqrt{\nu}) = z\tan\sigma$. Thus the result of the integral is

$$I = \int_{-1}^{+1} dt\, H(|\delta_t| - z)\, \ln\left(\frac{|\delta_t| + \sqrt{\delta_t^2 - z^2}}{|\delta_t| - \sqrt{\delta_t^2 - z^2}}\right)$$

$$I = \left[\ln\left(\frac{1+T}{1-T}\right) - 2T\right].$$

Where $T = \sqrt{1 - z^2/y^2}\, H(1 - z/y)$.

Final Gap Equation

$$\ln\left(\frac{z}{z_0}\right) + \frac{\widehat{\Theta}^2 y^2}{24}\left[2\ln\left(\frac{z}{8}\right) + \frac{16}{3}\right] + \frac{1}{2}\ln\left(\frac{1+T}{1-T}\right)\left(1 - \frac{\widehat{\Theta}^2 y^2}{12}\right)$$
$$-T\left(1 - \frac{\widehat{\Theta}^2 y^2}{8}\right) - \frac{\widehat{\Theta}^2 y^2}{24}\tan^{-1}\left(\frac{T}{\sqrt{1-T^2}}\right) = 0.$$

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Thermodynamic Potential

$$\Omega = \frac{z^2}{2}\left[\ln\left(\frac{z}{z_0}\right) - \frac{1}{2}\right] + \frac{z^2}{2}\left[\ln\left(\frac{1+T}{1-T}\right) - T\right] + \frac{y^2}{6}(1-T^3)$$
$$+\frac{\widehat{\Theta}^2 y^2 z^2}{48}\left[\ln\left(\frac{z^2}{64}\right) + \frac{16}{3}\right].$$

Results

We can expand the potential in z of the form $\Omega = \alpha z^2 + \beta z^4$. Where

$$\alpha = \left[-18 + 4y^2t^2 + 9\ln(4y^2/z_0^2)\right]/36$$
$$\beta = 1/(16y^2).$$

The best q dependance can be found by minimizing the above expression w.r.t. y^2 and we get $z^2/4$.