Ultracold atoms in the presence of a synthetic gauge field

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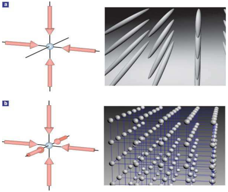
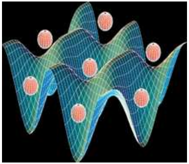
Outline

- Quantum Phase Transition of ultracold bosons: Superfluid to Mott insulator
- Mott-Superfluid transition in presence of synthetic gauge fields
- Bosons in artificial non-abelian gauge field (effective spin-orbit interaction).

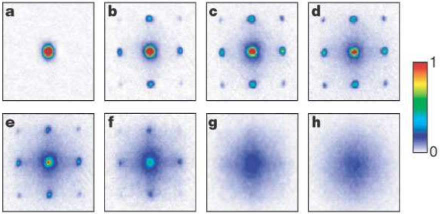
Ultracold atoms in an optical lattice

- periodic potential: $V(x) = V_0 \cos^2(kx)$
- V_0 can be controlled by the intensity of the laser beam.



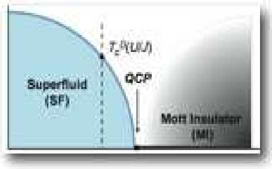


Quantum phase transition of Bosons in Optical Lattice: from Superfluid to Mott Insulator

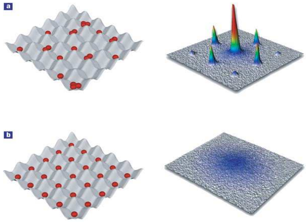


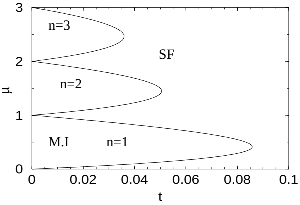
M. Greiner et. al
Nature415,39 (2002)

Increasing V (U/t)



Bose-Hubbard Model





- The Model:

$$\hat{H} = -t \sum_{i,\delta} \hat{a}_i^\dagger \hat{a}_{i+\delta} + \sum_i \left[\frac{U}{2} \hat{n}_i \left(\hat{n}_i - 1 \right) + \frac{m\omega^2}{2} \mathbf{r}_i^2 \hat{n}_i \right]$$

$$\hat{n}_i = a_i^\dagger a_i$$

Strong coupling limit: $U >> t$

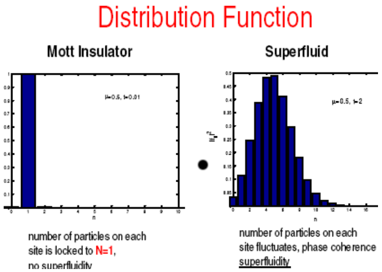
- $H_0 = \frac{U}{2}\sum_i \hat{n}_i(\hat{n}_i - 1) - \mu\hat{n}_i$
- Mott state : $|M.I >= \prod |n_0 >_i \qquad n_0 = 1, 2, 3....$
- Particle Hole excitation:

  $E_p = E(N+1) - E(N) = Un_0 - \mu$
  $E_h = E(N-1) - E(N) = \mu - U(n_0 - 1).$
- perturbation: $H_1 = -t\sum_{<ij>} a_i^\dagger a_j + h.c$
- Mean Field: $H_{MF} = -t(a^\dagger\psi + a\psi^*)$,
  Order parameter:
  $\psi = \sum_{nn} < a > \neq 0$ in SF, and $\psi = 0$ in M.I.
- second order perturbation :
  $E = E_{Mott} + r|\psi|^2 + O(|\psi|^4)$

  For $r < 0$ SF phase appears.

Gutzwiller's variational wave-function

- Product wave function: $|\Psi> = \prod_i \sum_n f_i^n |n>$
- In M.I $f_i^n = \delta_{n,n_0}$ In deep SF $|\Psi>$ becomes a coherent state and $a \to \phi$.
- In deep SF phase, the system can be described by DNLS (GP equation) of the classical fields ϕ.



D. L. Kovrizhin, G. V. Pai, S. Sinha, E.P.L 72, 162 (2005), cond-mat 07072937.

- Excitation Energies: Linearized time-dependent Gutzwiller Method
  Minimization of action: $S =< \Psi(f_i(t))|i\partial_t - H|\Psi(f_i)>$
- Gapped Particle, Hole excitation in Mott phase:
  $\varepsilon_{p(h)} = \sqrt{\frac{U^2}{4} + \frac{\epsilon_{\mathbf{k}}^2}{4} + \epsilon_{\mathbf{k}} U(n_0 + \frac{1}{2})} \pm \left[U(n_0 - \frac{1}{2}) - \mu + \frac{\epsilon_{\mathbf{k}}}{2}\right]$
  .
- Gapless sound mode in SF phase: $\varepsilon(k) = c_s k$
- sound velocity : $c_s = t\sqrt{d}\cos\theta\sqrt{(\alpha^2\cos^2\theta - 1)/2}$
  $\alpha = (\sqrt{n_0} + \sqrt{n_0 + 1})^2 = U/(2td\cos 2\theta)$
- Amplitude mode: $\omega(\mathbf{k}) = \sqrt{\Delta^2 + c^2\mathbf{k}^2}$

Ultracold atoms in artificial magnetic field

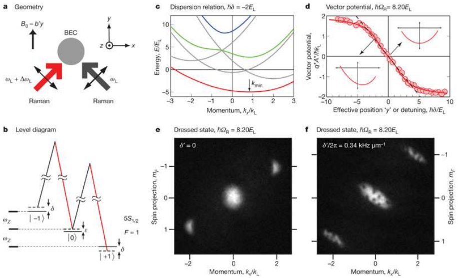
- Rotating condensate in 2D:

$$H = \frac{\vec{P}^2}{2m} + \frac{1}{2}m\omega^2 r^2 - \Omega \hat{L}_z.$$
$$= \frac{1}{2m}(\vec{P} - \vec{A})^2 + \frac{1}{2}m(\omega^2 - \Omega^2)r^2.$$

- $\vec{A} = (m\Omega y, -m\Omega x, 0).$

- Now it is possible to generate synthetic gauge fields:

Y.J. Lin et. al. Nature 462, 628 (2009). P.R.L 102, 130401 (2009).



- Using two laser beams and $F = 1$ state of ^{87}Rb atoms it is possible to generate single particle dispersion $E(k) \approx \frac{\hbar^2}{2m^*}(k_x - K_L(y))^2$.
- Effective gauge field : $A_x^* = K_L(y)$

SF-MI transition in the presence of a magnetic field

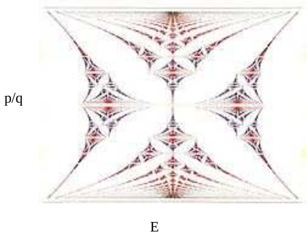
- Bose-Hubbard model:
  $\mathcal{H} = \sum_{i,j} J_{ij} b_i^\dagger b_j + \sum_i [-\mu \hat{n}_i + \frac{U}{2} \hat{n}_i (\hat{n}_i - 1)]$.
- Peierls substitution: $J_{ij} = -J \exp(-iq^* \int_i^j \vec{A}^* \cdot \vec{dl}/\hbar c)$
- In Landau gauge: $\vec{A}^* = B^*(0, x)$
- $\Phi = \frac{B^*}{a^2} = 2\pi p/q$ flux quanta passing through each lattice plaquette.

$$J_{x,x+1} = J \text{ and } J_{(x,y),(x,y+1)} = Je^{ix2\pi p/q}$$

S. Sinha, K. Sengupta, EPL (93) ,30005 (2011)

Single particle spectrum: Hofstadter's butterfly

- Wave function: $\Psi(x,y) = e^{\imath k_y y} g(x)$
  $-(g(x+1) + g(x-1)) - 2\cos(\phi x - k_y) g(x) = \epsilon g(x)$
- $\epsilon(k)$ forms q bands, with $-\pi/q \leq k_x \leq \pi/q$, $-\pi \leq k_y \leq \pi$.

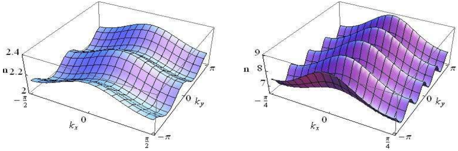


Mott Phase

- Particle hole excitation:
  $E_q^{\alpha\pm}(\mathbf{k}) = -\mu + U(n_0 - 1/2) + \epsilon_q^\alpha(\mathbf{k})/2 \pm \sqrt{\epsilon_q^\alpha(\mathbf{k})^2 + 4\epsilon_q^\alpha(\mathbf{k})Un_0 + U^2}/2$

  where $\epsilon_q^\alpha(k)$ are single particle energies of Hofstadter problem.

- Momentum Distribution for $q = 2, 4$



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Effective action: Strong coupling expansion

- Hubbard-Stratonovitch transformation:
  $Z = \int D\psi^* D\psi e^{-S_{eff}[\psi^*,\psi]}$

  $$e^{-S_{eff}} = \int Da^* Da \, exp[-\int_0^\beta d\tau \sum_i a_i^* \partial_\tau a_i + H[a^*, a, \psi^*, \psi]]$$

  $H = \sum_i \left[\frac{U}{2} n_i(n_i - 1) - \mu n_i - a_i^\dagger \psi_i - a_i \psi_i^*\right] + \sum_{i,j} \psi_i^* t_{ij}^{-1} \psi_j$.

  Effective action: $S_{eff} \approx S_0 + S_2$

  $$\begin{aligned} S_0 &= \int_{\mathbf{k}} \psi^*(i\omega_n, \mathbf{k})[-G_0^{-1}(i\omega_n) + \epsilon(\mathbf{k})]\psi(i\omega_n, \mathbf{k}), \\ S_1 &= g/2 \int_0^\beta d\tau \int d^2 r |\psi|^4. \end{aligned}$$

  Green's function:
  $G_0^{-1}(\omega) = (\omega - E_p)(\omega + E_h)/(\omega + U + \mu)$

- Near Mott phase boundary modes become soft at q momentum $Q^{\alpha} = (0, 2\pi\alpha/q)$ where $\alpha = 0, 1...q-1$.
- Landau-Ginzburg theory to be constructed out of q low-energy fluctuating fields $\phi^{\alpha}(r,t)$ around these minima:
  $\psi_q(r,t) = \sum_{\alpha=0}^{q-1} \chi_q^{\alpha}\phi^{\alpha}(r,t)$. where χ_q^{α} are eigenvectors of $J_q(\mathbf{Q}^{\alpha})$.
- Effective action:
  $\mathcal{S} = \int d^2 r dt \left[\sum_{\alpha} \phi_{\alpha}^{*}(k_0\partial_t^2 - ik_1\partial_t + r - v_q^2\nabla^2)\phi_{\alpha} + L_4(\phi_{\alpha})\right]$

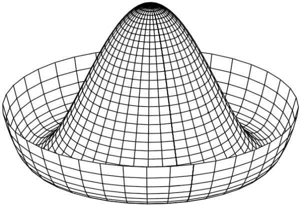
Effective action for $q = 2, 3$

- $\mathcal{L}_4(q=2) = \frac{1}{8}[3g(|\xi^0|^2 + |\xi^1|^2)^2 - g(|\xi^0|^2 - |\xi^1|^2)^2]$
- $S_2^{\prime\, q=3}$ turns out to be $O(3)$ symmetric: $\mathcal{L}_4(q=3) \sim (\sum_{\alpha=0..2} |\xi^\alpha(\mathbf{r}, t)|^2)^2$.

Higgs and Goldstone modes

$$L_H = \frac{1}{2}|\partial\phi|^2 - V(\phi)$$

- The classical potential: Landau-Ginzburg free energy functional: $V(\phi) = m^2|\phi|^2 + \lambda^4|\phi|^4$



- After symmetry breaking, two modes apear:
  Massless Goldstone mode: $\omega \sim |k|$
  Massive (gapped) Higgs mode: $\omega = \sqrt{c_s^2 k^2 + m^2}$

- For $q = 2$ corresponding to condensate field we find one Goldstone and one Higgs mode:

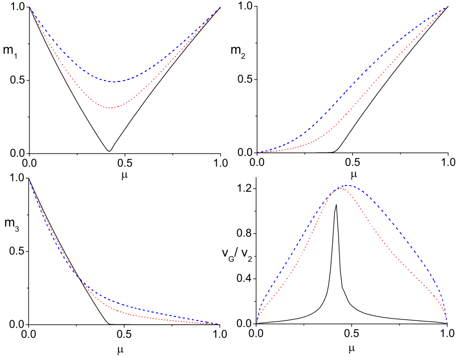
$$\omega^2 = (v_q^2 k^2 - r)/k_0 + \frac{1}{2}(\frac{k_1}{k_0})^2 \pm \sqrt{\frac{(-r + v_q^2 k^2)}{k_0}(\frac{k_1}{k_0})^2 + \frac{1}{4}(\frac{k_1}{k_0})^4 + \frac{r^2}{k_0^2}}$$

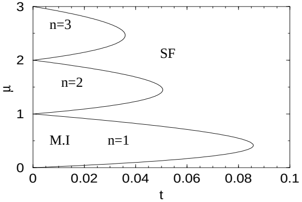
- For non-condensate mode:

$$\omega = \left[\frac{1}{2}(\pm\frac{k_1}{k_0}) + \sqrt{\frac{1}{4}(\frac{k_1}{k_0})^2 + \frac{v_q^2 k^2}{k_0} - r/2}\right]$$

- For $q = 3$, corresponding to condensate mode we find similar sound mode and Higgs mode as $q = 2$.
- Corresponding to two noncondensate fields: Two gapless modes $\omega \sim k^2$ and two gapped modes.

- Higgs mass and sound velocity:





BEC with strong spin-orbit interaction

- Bosons in presence of artificial gauge fields:

$$H = \int d^2\vec{r} \left\{ \psi^\dagger \left[\frac{1}{2M} (\vec{p} - \vec{A})^2 + \frac{1}{2} m\omega^2 r^2 \right] \psi + \frac{g}{2} (\psi^\dagger \psi)^2 \right.$$

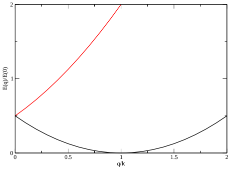
  It is possible to generate $SU(2)$ gauge fields: $(A_x, A_y) = \hbar\kappa(\sigma_x, \sigma_y)$.

- Effective Spin-Orbit interaction: $H = H_0 + H_I$

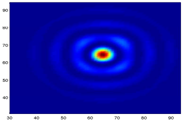
  $H_I = \frac{\hbar\kappa}{M} \vec{\sigma}.\vec{P}$

  in collaboration with L. Santos (Hannover)

Single particle states and Kramer's degeneracy



- Single particle energies: $E(q) = \frac{\hbar^2}{2m}(|q| - \kappa)^2$
  Wavefunction: $\psi_k = e^{\imath \vec{k}.\vec{r}}\,(1,\; e^{\imath\phi_k})$
  $\psi_{-k} = \imath\sigma_y C\psi_k$
- States in harmonic trap:
  $(\phi_1, \phi_2) = \sqrt{N}e^{-r^2/2}(J_l(\kappa r)e^{\imath l\theta}, \imath J_{l+1}(\kappa r)e^{\imath(l+1)\theta})$
  $(\phi_1, \phi_2)^T =$
  $-\sqrt{N}e^{-r^2/2}(\imath J_{l+1}(\kappa r)e^{-\imath(l+1)\theta}, J_l(\kappa r)e^{-\imath l\theta})$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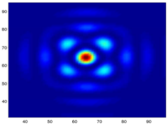
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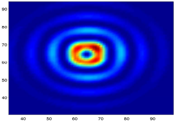
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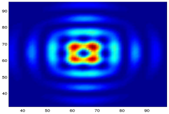
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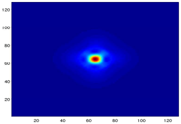
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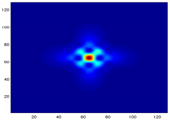
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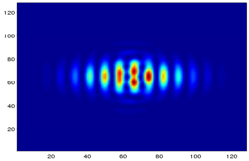
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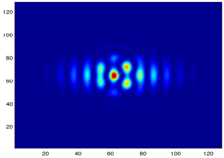
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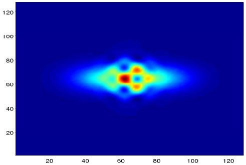
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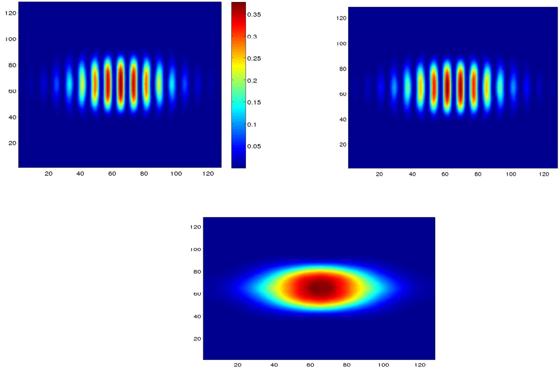
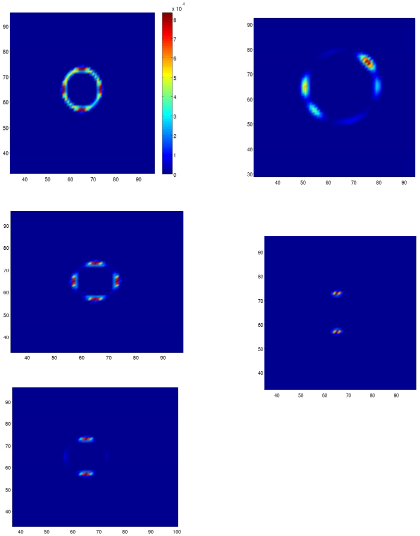
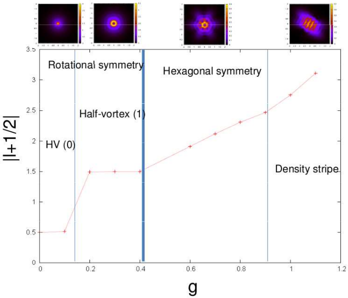
Density wave state



Momentum distribution



Phase diagram



Stability of SDW state: Landau-Ginzburg theory

$$E = E_{SDW} + \Delta^2 \left[\frac{gN}{2\pi l_x l_y} - m\omega^2 \right] + O(\Delta^4)$$

where Δ^2 is width of the momentum distribution of SDW around diagonally opposite points $\vec{k} = \vec{\kappa}, -\vec{\kappa}$ on the momentum ring.

Outlook

- Cold atoms can be used as Hubbard toolbox.
- It is interesting to study excitations of cold atoms coupled to non-abelian gauge fields.
- Quantum Hall states of bosons can be studied.