Signatures of the κ-Minkowski Spacetime at Low Energies

E. Harikumar[1]

School of Physics
University of Hyderabad
Hyderabad

August 2011

[1] harisp@uohyd.ernet.in

Motivations....

- In order to probe spacetime at the Planck scale l_P, the Compton wavelength $\frac{\hbar}{Mc}$ of the probe must fulfill

$$\frac{\hbar}{Mc} \leq l_P \quad or \quad M \geq \frac{\hbar}{l_P c} \simeq \text{Planck mass.}$$

- Such high mass in the small volume l_P^3 will strongly affect gravity and can lead to black holes and their horizons to form. This suggests a fundamental length limiting spatial localization.
- Similar arguments holds time localization.
- Observation of very short time scales requires very high energies. They can produce black holes and black hole horizons will then limit spatial resolution suggesting

$$\Delta t \, \Delta|\overrightarrow{x}| \;\geq\; l_P^2 \, l_P \;=\; \text{a fundamental length.}$$

- The Noncommutative space-time models above spacetime uncertainties. (Doplicher, Fredenhagen, Roberts)

Motivations....

- All different approaches to study quantum gravity like String Theory, Loop Gravity, Causal set approach, etc do predict the existence of a minimum length scale.
- Thus there are compelling reasons to expect the existence of a fundamental length scale and we need to incorporate this length scale in the discussions of quantum gravity.
- Quantum gravity can be, possibly modeled using non-commutative space-time, which will naturally bring in a length scale into the discussion.

- But if such a fundamental length scale exist, it will be in direct conflict with Special Theory of Relativity.
- This led to study of possible modifications of STR. Now we have a mathematically consistant theory of relativity. This modified relativity principle, has two fumdamental constants, velocity of light c and a fumdamental length scale..**D**eformed **S**pecial **R**elativity.
- Now the transformations that leave physics invariant are Deformed Lorentz transformations,
- and symmetry group is Deformed Lorentz (or Poincare) group.

κ-Deformation

- The fumdamental length scale modifies the Lorentz/Poincare transformations.
- Poincare transformations are generated by P_μ, $M_{0i} = N_i$, and $M_{ij} = \epsilon_{ijk} J_k$
- Poincare algebra $(M_i = J_i)$

$$[P_\mu, P_\nu] = 0, \quad [M_i, P_\mu] = i\epsilon_{i\mu j} P_j, \quad [N_i, P_0] = iP_i$$

$$[N_i, P_j] = i\delta_{ij} P_0$$

$$[M_{\mu\nu}, M_{\alpha\beta}] = i(\eta_{\mu\beta} M_{\nu\alpha} - \eta_{\mu\alpha} M_{\nu\beta} + \eta_{\nu\alpha} M_{\mu\beta} - \eta_{\nu\beta} M_{\mu\alpha})$$

- $P_\mu P^\mu$ is the Casimir of Poincare algebra
- We can label representations of Poincare algebra using the value for the Casimir
  $P_\mu P^\mu = m^2c^2$ is the Energy-Momentum relation,
  $E^2 = p^2c^2 + m^2c^4$
- The quadratic Casimir of the underlying symmetry algebra gives the dispersion relation
- The existence of a fumdamental length scale Deforms the Poincare algebra.
- The deformed algebra relevant for us is the kappa-Poincare algebra.
- Underlying space-time is κ-Minkowski spacetime;

$$[\hat{x}_i, \hat{x}_j] = 0, \quad [\hat{x}_0, \hat{x}_i] = ia\hat{x}_i$$

Symmetry algebra of κ-spacetime

- The symmetry algebra of this spacetime is κ-Poincare algebra

$$[M_{\mu\nu}, M_{\alpha\beta}] = i(\eta_{\mu\beta}M_{\nu\alpha} - \eta_{\mu\alpha}M_{\nu\beta} + \eta_{\nu\alpha}M_{\mu\beta} - \eta_{\nu\beta}M_{\mu\alpha})$$

$$[N_i, P_j] = i\delta_{ij}\left(\frac{1}{2a}(1 - e^{-2aP_0}) + \frac{a}{2}\vec{P}^2\right) - iaP_iP_j$$

with Casimir $m^2 = (\frac{2}{a}sinh(\frac{aP_o}{2}))^2 - \vec{P}^2e^{aP_0}$

- There are different approaches to construct field theory on k-spacetime.
- Using fields which are functions of $\hat{x}_\mu$ and defining the action which is invariant under κ-Poincare algebra.
- Map κ-spacetime coordinates and their functions to commutative ones and work with these commutative functions.
    - We take the second approach

K-spacetime, ordering, Leibnitz rules

- We have $[\hat{x}_0, \hat{x}_i] = ia\hat{x}_i, \quad [\hat{x}_i, \hat{x}_j] = 0$
- $\hat{x}_\mu = x_\alpha \Phi_{\alpha\mu}(\partial)$ This defines a unique mapping of functions on k-spacetime to that on commutative space time
- Imposing

$$[\partial_i, \hat{x}_j] = \delta_{ij}\varphi(A), \quad [\partial_i, \hat{x}_0] = ia\partial_i\gamma(A)$$

$$[\partial_0, \hat{x}_i] = 0, \quad [\partial_0, \hat{x}_0] = 1,$$

with $A = ia\partial_0$, we get from $\hat{x}_\mu = x_\alpha \Phi_{\alpha\mu}(\partial)$

K-spacetime, ordering, Leibnitz rules

- $$\hat{x}_i = x_i \varphi(A)$$
  $$\hat{x}_0 = x_0 \psi(A) + i a x_i \partial_i \gamma(A)$$
- from the commutators we get $\frac{\varphi'}{\varphi}\psi = \gamma - 1$
  $(\ \varphi(0) = 1, \psi(0) = 1, \gamma(0) = \varphi'(0) + 1)$
- Leibnitz rule for ∂_i is modified
  $$\Delta_\varphi(\partial_i) = \partial_i^x \frac{\varphi(A_x + A_y)}{\varphi(A_x)} + \partial_i^y \frac{\varphi(A_x + A_y)}{\varphi(A_y)}$$
- $$\Delta_\varphi(\partial_0) = \partial_0 \otimes I + I \otimes \partial_0 = \partial_0^x I^y + I^x \partial_0^y$$

k-Poincare algebra, Casimir and Dispersion relation

- Define Dirac derivatives and their algebra:
  $[M_{\mu\nu}, \hat{x}_\lambda] = \hat{x}_\mu \delta_{\nu\lambda} - \hat{x}_\nu \delta_{\nu\lambda} - ia_\mu M_{\nu\lambda} - ia_\nu M_{\mu\lambda}$
  $$[M_{\mu\nu}, D_\lambda] = \delta_{\nu\lambda} D_\mu - \delta_{\mu\lambda} D_\nu$$
  $$[D_\mu, D_\nu] = 0$$
  $$[D_\mu, \hat{x}_\nu] = \delta_{\mu\nu}\sqrt{1 - a^2 D_\alpha D_\alpha} + ia_0(\delta_{\mu 0} D_\nu - \delta_{\mu\nu} D_0)$$
  $$D_0 = -i\partial_0 \frac{sinhA}{A} - ia(\partial_i)^2 \frac{e^{-A}}{2\varphi^2}; \qquad D_i = \partial_i \frac{e^{-A}}{\varphi}$$
- $$[M_{\mu\nu}, \Box] = 0, [\Box, \hat{x}_\mu] = 2D_\mu$$
  $$\Box = (\partial_i)^2 \frac{e^{-A}}{\varphi^2} + 2\partial_0^2(1 - coshA)/A^2$$

k-Poincare algebra, Casimir and Dispersion relation

- The Casimir

$$D_\mu D_\mu = \Box(1 - \frac{a^2}{4}\Box) \quad \text{quartic}$$

- Dispersion relation ($E^2 = P^2c^2 + m^2c^4$, $p_i = m\dot{x}_i$)

$$\frac{4}{a^2}\sinh^2(\frac{aE}{2c}) - p_i p_i \frac{e^{-a\frac{E}{c}}}{\varphi^2(a\frac{E}{c})} - m^2c^2$$
$$+\frac{a^2}{4}\left[\frac{4}{a^2}\sinh^2(\frac{aE}{2c}) - p_i p_i \frac{e^{-a\frac{E}{c}}}{\varphi^2(a\frac{E}{c})}\right]^2 = 0$$

- We have constructed scalar field theory on κ-spacetime.
- Investigated the implication of κ-deformation on the flip operators
  i.e., $\tau : \Phi(x)\Phi(y) \to \Phi(y)\Phi(x)$ The statistics is modified.

Phys.Rev.D80:025014,2009;Phys.Rev.D77:105010,2008 T R Govindarajan, K S Gupta, EH, D. Meljanac, S.Meljanac

Questions

- There may exist effects/signals reminiscent of underlying NC structure at low energies. It is important to look for such signals.
- Our strategy is to start with the deformed disperion relation, expand in powers of a, keep only terms up to first order in a.
- See whether there are significant effects which are affected by this 1st order deformation.

Dirac Equation on κ Spacetime

- Dispersion relation

$$\frac{a^2}{4}Sinh^2(\frac{ap_0}{2}) - p_i^2\frac{e^{-ap_0}}{\varphi^2(ap_0)} - m^2$$
$$-\frac{a^2}{4}\left[\frac{a^2}{4}Sinh^2(\frac{ap_0}{2}) - p_i^2\frac{e^{-ap_0}}{\varphi^2(ap_0)}\right]^2 = 0$$

- Dirac Derivatives are

$$\begin{aligned} D_i &= \partial_i\frac{e^{-A}}{\varphi} \\ D_0 &= \partial_0\frac{sinhA}{A} + ia\nabla^2\frac{e^{-A}}{2\varphi^2}, \end{aligned}$$

- Dirac Equation

$$(\gamma^0 D_0 + \gamma^i D_i + \frac{mc}{\hbar})\Psi = 0$$

Parity

- Under parity, $P : x \rightarrow -x, \;\; P : t \rightarrow t,$
- $P : D_i \rightarrow -D_i, \;\; P : D_0 \rightarrow D_0$
- $(\gamma^0 D_0 - \gamma^i D_i + mc\hbar^{-1})\Psi(-x, t) = 0$
- $\mathcal{P}\Psi = \gamma_0 P\Psi(x, t)$ is a solution if $\Psi(x, t)$ is a solution to Dirac Eqn.
    - Parity is a symmetry of the κ-Dirac equation

Time reversal

- $\mathcal{T}: x \rightarrow x, \;\; \mathcal{T}: t \rightarrow -t,$
- $\mathcal{T}: D_i \rightarrow D_i, \;\; \mathcal{T}: D_0 \rightarrow \tilde{D}_0$ where

$$\tilde{D}_0 = -\frac{i}{a} sinh(-ia\partial_0) + \frac{ia}{2}\nabla^2 e^{+ia\partial_0}$$

- $i\hbar \tilde{D}_0 \Psi(x,-t) = H\Psi(x,-t)$
- $\mathcal{T}\Psi^*(x,t) = -i\alpha_1\alpha_3 T\Psi^*$ is a solution if $\Psi(x,t)$ is a solution to Dirac Eqn.
  - Time Reversal is a symmetry of the κ-Dirac equation

Charge Conjugation

- Minimaly coupled Dirac Eqn.

$$i\hbar\left(\frac{i}{a}sinh[a(p_0-eA_0)]-\frac{ia}{2\hbar^2}(\vec{p}-e\vec{A})^2e^{a(p_0-eA_0)}\right)\Psi$$
$$=\left(\vec{\alpha}\cdot(\vec{p}-e\vec{A})+\beta mc\right)\Psi$$

- we get

$$i\hbar\left(\frac{i}{a}sinh[a(p_0+eA_0)]+\frac{ia}{2\hbar^2}(\vec{p}+e\vec{A})^2e^{-a(p_0+eA_0)}\right)\Psi^*$$
$$=\left(\vec{\alpha}^*\cdot(\vec{p}+e\vec{A})-\beta^* mc\right)\Psi^*$$

- Similarity transformation by $C=i\beta\alpha_2$ st $C\alpha^*C^{-1}=\alpha$, $C\beta^*C^{-1}=-\beta$ takes RHS of 2nd to 1st. Similarity transformation will not change the sign of the second term as well as the sign of the exponential.
- Charge conjugation is NOT a symmetry . Particle and anti-particle have different Equations

H- Atom Spectrum

- κ-Dirac equation for Hydrogen atom (valid up to 1st order in a) is

$$i\hbar\partial_t\Psi = \left[-i\hbar c\vec{\alpha}\cdot\vec{\nabla} + mc^2\beta + V(r) + \frac{ac\hbar}{2}\vec{\nabla}^2\right]\Psi$$

  where $V(r) = -\frac{Ze^2}{4\pi\epsilon_0 r}$
- first order perturbation to $1S_{\frac{1}{2}}$ and $2S_{\frac{1}{2}}$ are

$$\Delta E_1 = -0.10256a\ J, \qquad \Delta E_2 = -236.32a\ J$$

- We get

$$|\frac{\Delta E_1 - \Delta E_2}{E_1}| = a(2.89\times 10^{15})m^{-1} \qquad (1)$$

  The frequency of $1S - 2S$ is a now known to an accuracy of 10^{-14}. Thus, we get $a2.89\times 10^{15} < 10^{-14}m$ implying

$$a < 10^{-29}m$$

- In the NR limit, we get

$$E'U = \left[-\frac{\hbar^2}{2m}\nabla^2 + V + \hbar^2\frac{(E'-V)}{4m^2c^2}\nabla^2 + \frac{1}{2m^2c^2}\frac{1}{r}\frac{dV}{dr}(L\cdot S)\right]U$$
$$+\left[-\frac{\hbar^2}{4m^2c^2}\frac{dV}{dr}\frac{\partial}{\partial r} + \frac{ac\hbar}{2}\nabla^2 - \frac{a\hbar^3}{8m^2c}\nabla^4\right]U$$

- Energy Eigenvalue

$$E_n = E_n^0\left[1 + \left(\frac{Z\alpha}{n}\right)^2\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right)\right]$$
$$+E_n^0\frac{acm}{\hbar}\left[1 - \left(\frac{Z\alpha}{n}\right)^2\left(\frac{3}{4} - \frac{n}{l+\frac{1}{2}}\right)\right]$$

κ-deformation breaks the degeneracy in the orbital quantum number l. But do not lift the degeneracy in m.

H- Atom Spectrum

- For $n = 1$, the ratio of the shift in Rydberg energy to the unperturbed one

$$|\frac{\Delta E}{E}| = \frac{a}{2} \times 10^{11} m^{-1}$$

- accuracy in the measurement of Rydberg energy is 10^{-8}. Hence.

$$a < 10^{-19} m$$

  LHC scale!
- Shift in Rydberg Energy is distinctive signal of κ-deformation

Deformed Kepler Problem

- Dispresion relation

$$\frac{4}{a^2}\sinh^2\left(\frac{aE}{2c}\right) - p_i p_i \frac{e^{-a\frac{E}{c}}}{\varphi^2(a\frac{E}{c})} - m^2c^2$$
$$+\frac{a^2}{4}\left[\frac{4}{a^2}\sinh^2\left(\frac{aE}{2c}\right) - p_i p_i \frac{e^{-a\frac{E}{c}}}{\varphi^2(a\frac{E}{c})}\right]^2 = 0$$

- With $\varphi(A) = e^{-A}$, in the NR limit, we find

$$H = \frac{(1-acm)}{2m}\,\vec{p}\cdot\vec{p}$$

- Since in our realisation, $\hat{x}_0 \to x_0, \quad \hat{x}_i \to x_i\varphi(A)$, we get

$$\frac{1}{\sqrt{\hat{x}_i\hat{x}_i}} \to \frac{1}{r} - \frac{a\dot{r}}{2cr^2}$$

- $$H = \frac{(1 - acm)}{2m}\,\vec{p}\cdot\vec{p} - \frac{K}{r} + \frac{Ka}{2mcr^3}(x\cdot p)$$
- Radial Eqn

$$\ddot{r} - r\dot{\phi}^2 = -\frac{K}{m}(1 - acm)\frac{1}{r^2}$$

- Polar eqn. (with $m_{eff} = m/(1 - acm)$)

$$\frac{d}{dt}(m_{eff}r^2\dot{\phi}) = 0$$

Pioneer anomaly

- Radial Eqn. $\ddot{r} - r\dot{\phi}^2 = -\frac{K}{m}(1 - acm)\frac{1}{r^2}$
- change in the magnitude of the radial force. This modification depends on the distance from sun in addition to Kac.
- an acceleration different from that given by the Newton's law of gravitation
- magnitude and direction will depend on the deformation parameter a as well as on r.
- The Pioneer anomaly: constant acceleration of $8.5 \times 10^{-10} m/s^2$ directed towards the sun, unaccounted by the Newton's law.

- This additional force is not constant, but its variation with respect to r is small, since a is expected to be very small.
- for the acceleration to be pointing towards sun, a should be negative.
- By approximating this additional force to be a constant (as far as Pioneer measurements are concerned), we can get a bound on $|a| \leq 10^{-53}m$.
- Clearly, more sensitive measurements are needed. If Pioneer anomaly is still turn out to be a constant force, and if κ deformation is also a fact, it leads to more questions!

Violation of Equivalence Principle

- EOM: $m_{eff}\ddot{x}_i = -\frac{\partial V}{\partial x_i}$
- Combining with Newtons Law of gravitation $\mu_{eff}\ddot{x} = -G\frac{m_g M_g}{r^2}$ we find

$$\ddot{x} = -G\frac{M_g}{r^2}\frac{m_g}{m}(1 - amc)$$

- deviation in the ratio of gravitational to inertial mass is

$$\delta(\frac{m_g}{m}) = -\frac{amc}{\hbar}$$

- EP violation is not unifrom, but depends on the inertial mass of the body
- $|a| \leq 10^{-55}m$

(covariant generalisation of) Feynman's approach

- relativistic particle in 4-dimensions is described by $x^\mu(\tau)$, τ is just a parameter and not the proper time.
- $[x^\mu, x^\nu] = 0$,
  Feynman postulated $[x^\mu, \dot{x}^\nu] = \frac{i\hbar}{m}\eta^{\mu\nu}$
- Newtons Eqn. of motion is assumed to hold. i.e., $m\ddot{x}^\mu = F^\mu(x, \dot{x})$
- $[A, B] = -[B, A]$,
  $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$
- the Leibniz rules

$$
\begin{aligned}
[A, BC] &= [A, B]C + B[A, C], \\
\frac{d}{dt}[A, B] &= [\frac{A}{dt}, B] + [A, \frac{dB}{dt}]
\end{aligned}
$$

- Leibniz rules are not automatically satisfied by Poisson brackets (need canonical equations of motion)

(covariant generalisation of) Feynman's approach

- Differentiating Feynman bracket gives
  $[\dot{x}^\mu, \dot{x}^\nu] = -[x^\mu, \ddot{x}^\nu] = \frac{iq\hbar}{m^2} F^{\mu\nu}$
  $F^{\mu\nu}$ is an arbitrary rank-2, anti-symmetric tensor
- Feynman derived homogeneous Maxwell eqn. by repeted use of Jacobi identities involving coordinates and momenta (this would also identify $F^{\mu\nu}$ with Maxwell's field strength)
  $\partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} + \partial^\lambda F^{\nu\mu} = 0$
- Force equation is also derived $F^\mu(x, \dot{x}) = G^\mu + qF^{\mu\nu}\dot{x}_\nu$
- Inhomogeneous Eqn. $\partial_\mu F^{\mu\nu} = J^\nu$ is taken as defining relation for the current (This can also be derived).

κ-Lorentz Eqn

- Start with $[\hat{x}^\mu, \hat{x}^\nu] = ic^{\mu\nu}{}_\lambda \hat{x}^\lambda$
- adpat the approach to $kappa$-spacetime
- Force equation and Maxwell's eqns. are derived
- Using $F^\mu = m_{eff}\ddot{x}^\mu$, we get

$$F(x, \dot{x})^\mu = G(x)^\mu + qF^\mu{}_\nu \dot{x}^\nu$$

- $G^\mu = \partial^\mu \Phi + O(a)$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + O(a)$

κ-Maxwell's Eqn

- $\frac{dx^i}{d\tau} = v^i$,

$$\vec{\nabla} \cdot \vec{B} + \frac{ma}{\hbar} \vec{v} \cdot \partial_0 \vec{B} = 0,$$

$$\partial_0 \vec{B} + \vec{\nabla} \times \vec{E} + \frac{ma}{\hbar} \left[v^i \partial_i \vec{B} + \vec{v} \times \partial_0 \vec{E} \right] = 0$$

$$\vec{\nabla} \cdot \vec{E} + \frac{ma}{\hbar} \vec{v} \cdot \partial_0 \vec{E} = \rho_e$$

$$\partial_0 \vec{E} - \vec{\nabla} \times \vec{B} + \frac{ma}{\hbar} \left[v^i \partial_i \vec{E} - \vec{v} \times \partial_0 \vec{B} \right] = -\vec{j}_e$$

- in the limit $a \to 0$, we get the Maxwell's Eqns. in commutative spacetime.
- E-M duality:

$$\vec{E} \to \vec{B}, \vec{B} \to -\vec{E}$$

$$\rho_e, j_e \longrightarrow \rho_{mag}, j_{mag}$$

$$\rho_{mag}, j_{mag} \longrightarrow -\rho_e, -j_e$$

Conclusions

- We have constructed certain physical models and analysed implications of κ deformation
- We have seen distigushing signals from κ-deformation.

κ-Dirac equation, EH, M. Sivakumar and N. Srinivas; Mod. Phys. Lett. **A26** (2011) 1103

Newton's Equation on the kappa space-time and the Kepler problem, EH and A. K. Kapoor, Mod.Phys. Lett. **A25** (2011) 2991)

Maxwell's equations on the κ-Minkowski spacetime and Electric-Magnetic duality, EH,Europhys.Lett. **90** (2010)21001.

Electrodynamics on κ-Minkowski space-time, EH, T. Juric and S. Meljanac, arXiv:1107.3936