Generation of all possible Quantum Optical Coherent States

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Outline

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- A Generalized Master Equation for Janus Faced Coherent States
- Thermofield Dynamics
- Solution of Master equation
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- Conclusion
- Entanglement in Non Linear Medium.
- Conclusion

This work is in collaboration with Prof V Srinivasan

Coherent State

- Coherent states $\longrightarrow$ quantum optics.
- Saturates the minimum uncertainty principle exactly
- It is taken as classical state.
- Any state which deviates from it is a non-classical state.
- The operator $\hat{a}|0\rangle = 0$ and satisfies the eigenvalue equation $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, where α is the complex eigenvalue and the eigenstate $|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha\hat{a}^\dagger}|0\rangle$ is called the coherent state.
- $a^\dagger$ is canonical conjugate of a
- Satisfying $[\hat{a}, \hat{a^\dagger}] = 1$.

Janus Faced Coherent States

- An operator $\mathcal{F}$ annihilates the vacuum $\mathcal{F}|0\rangle = 0$
- $\mathcal{G}^\dagger$ is canonical conjugate of $\mathcal{F}$
- Satisfying $[\mathcal{F}, \mathcal{G}^\dagger] = 1$.
- Can one construct a coherent states?
- Yes
- One has two eigenstates, one for $\mathcal{F}$ and the other for $\mathcal{G}$.

Cont......

- $\mathcal{F}$ satisfies $\mathcal{F}|f\rangle_i = f|f\rangle_i$
- The eigenstate $|f\rangle_i = exp[f\mathcal{G}^\dagger]|0\rangle$.
- Similarly $\mathcal{G}$ satisfies $\mathcal{G}|g\rangle_i = g|g\rangle_i$
- The eigenstate $|g\rangle_i = exp[g\mathcal{F}^\dagger]|0\rangle$.
- These coherent states occur in pairs, hence the name Janus Faced coherent states.

P. Shanta, S. Chaturvedi, V. Srinivasan, G.S. Agarwal and C. L.Metha, 1994, Phy Rev Lett **72**, 1447.

Cont......

- The Caves-Schumaker state $|\alpha\rangle = exp\left[\alpha ab - \alpha^* a^\dagger b^\dagger\right]|0\rangle$ and the pair coherent state $ab|\zeta, q\rangle = \zeta|\zeta, q\rangle$, is one such pair.
- The Yuen state $|\alpha\rangle = exp\left[\alpha a^2 - \alpha^* a^{\dagger 2}\right]|0\rangle$ and the Cat state $|\alpha\rangle = -\frac{1}{\sqrt{2}}(|\alpha\rangle + |-\alpha\rangle)$ forms another pair.
- The operator $\mathcal{F}$ can annihilate more than one vacuum with corresponding $\mathcal{G}^\dagger$.

Pair Coherent States

- The master equation for the pair coherent state is

$$\frac{\partial}{\partial t}\rho = -ig\left(ab\rho - \rho ba + a^\dagger b^\dagger \rho - \rho b^\dagger a^\dagger\right) + \frac{\kappa}{2}\left(2ab\rho a^\dagger b^\dagger - a^\dagger b^\dagger ab\rho - \rho a^\dagger b^\dagger ab\right),$$

  here g and κ are the arbitrary parameters.

G.S. Agarwal,1986, Phy Rev Lett **57**, 827

A Generalized Master Equation for Janus Faced Coherent States

- The Caves-Schumaker state and the pair coherent state form a “Janus faced” pair.
- Can one write a master equation for other coherent states which form a “Janus faced” pair ?
- The answer to this question is affirmative.
- The generalized master equation

$$\begin{aligned}\frac{\partial}{\partial t}\rho &= -ig\left(\mathcal{F}\rho - \rho\mathcal{F} + \mathcal{F}^\dagger\rho - \rho\mathcal{F}^\dagger\right) \\ &+\frac{\kappa}{2}\left(2\mathcal{F}\rho\mathcal{F}^\dagger - \mathcal{F}^\dagger\mathcal{F}\rho - \rho\mathcal{F}^\dagger\mathcal{F}\right).\end{aligned}$$

Master Equation

- In quantum mechanics, unitary evolution of a closed physical system or pure state is described by the Schroedinger equation.
- In nature, most of the physical systems are not associated with a unitary evolution.
- The mixed states when appended with an environment evolve unitarily giving rise to dissipation.
- Evolution of these dissipative systems is described by a master equation.
- The master equation lies at the heart of quantum optics.
- The master equation is used to model decoherence, entanglement sudden death, quantum channel and found many applications in quantum optics.

The thermo field dynamics is a finite temperature field theory.

The salient features of this approach

- The solving of master equation is reduced to solving a Schroedinger equation, thus all the techniques available to solve the Schroedinger equation are applicable here.

S. Chaturvedi and V. Srinivasan, *J. Mod. Opt.* **38**, 777

Brief Description

- In TFD $|\rho^{\alpha}\rangle, 1/2 \leq \alpha \leq 1$, is a state vector in the extended Hilbert space $\mathcal{H} \otimes \mathcal{H}^{*}$.
- $\langle A\rangle = TrA\rho = \langle \rho^{1-\alpha}|A|\rho^{\alpha}\rangle$
- $|\rho^{\alpha}\rangle$ is given by $|\rho^{\alpha}\rangle = \hat{\rho}^{\alpha}|I\rangle$. and $\hat{\rho}^{\alpha} = \rho^{\alpha} \otimes I$,
- $|I\rangle$ is resolution of the identity

$$|I\rangle = \sum |n\rangle\langle n| = \sum |n\rangle \otimes |\tilde{n}\rangle \equiv \sum |n, \tilde{n}\rangle,$$

in terms of a complete orthonormal set $|n\rangle$ in $\mathcal{H}$.

- The creation and the annihilation

$$\begin{aligned} a|n,m\rangle &= \sqrt{n}|n-1,m\rangle, \\ a^\dagger|n,m\rangle &= \sqrt{n+1}|n+1,m\rangle, \\ \tilde{a}|n,m\rangle &= \sqrt{m}|n,m-1\rangle, \\ \tilde{a}^\dagger|n,m\rangle &= \sqrt{m+1}|n,m+1\rangle. \end{aligned}$$

- $a|I\rangle = \tilde{a}^\dagger|I\rangle,\ a^\dagger|I\rangle = \tilde{a}|I\rangle$
- Any operator $A|I\rangle = \tilde{A}^\dagger|I\rangle$
- The tilde conjugation rules are $a \to \tilde{a}, a^\dagger \to \tilde{a}^\dagger, \alpha \to \alpha^*$.
- This reflects the hermiticity property of the density operator.

Non Equilibrium Thermo Field Dynamics

- The non equilibrium thermo field dynamics is developed for $\alpha = 1$ representation.
- In this representation, $\langle A \rangle = \langle I | A | \rho \rangle = \langle A | \rho \rangle = Tr(A\rho)$.
- Given any master equation of the form

$$\frac{\partial}{\partial t}\rho(t) = \frac{-i}{\hbar}(H\rho - \rho H) + L\rho,$$

- One converts this into a problem in TFD by applying $|I\rangle$ from the right

$$\frac{\partial}{\partial t}|\rho(t)\rangle = -i\hat{H}|\rho\rangle$$

where

$$-i\hat{H} = i(H - \tilde{H}) + L.$$

In TFD $-i\hat{H}$ is tilden.

Example

- Consider a damped harmonic oscillator master equation

$$\begin{aligned}\frac{\partial}{\partial t}\rho(t) &= i\omega(a^\dagger a\rho - \rho a^\dagger a) \\ &+ \frac{\kappa(\bar{n}+1)}{2}\left(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a\right) \\ &+ \frac{\kappa\bar{n}}{2}\left(2a^\dagger\rho a - a^\dagger a\rho - \rho a^\dagger a\right)\end{aligned}$$

By applying $|I\rangle$ one goes over to TFD

$$\begin{aligned}\frac{\partial}{\partial t}|\rho(t)\rangle &= \Big(i\omega(a^\dagger a - \tilde{a}\tilde{a}^\dagger) \\ &+ \frac{\kappa(\bar{n}+1)}{2}\left(2a\tilde{a} - a^\dagger a - \tilde{a}\tilde{a}^\dagger\right) \\ &+ \frac{\kappa\bar{n}}{2}\left(2a^\dagger\tilde{a}^\dagger - aa^\dagger - \tilde{a}^\dagger\tilde{a}\right)\Big)|\rho(t)\rangle.\end{aligned}$$

Solution to Master Equation

- One goes over to TFD formalism by applying $|I\rangle$.
- The $-i\hat{H}$ for generalized master equation for Janus faced coherent states

$$
\begin{aligned}
-i\hat{H} &= [-ig\left(\mathcal{F} - \tilde{\mathcal{F}}^{\dagger} + \mathcal{F}^{\dagger} - \tilde{\mathcal{F}}\right) \\
&\quad + \frac{\kappa}{2}\left(2\mathcal{F}\tilde{\mathcal{F}} - \tilde{\mathcal{F}}\tilde{\mathcal{F}}^{\dagger} - \mathcal{F}^{\dagger}\mathcal{F}\right)].
\end{aligned}
$$

- The initial state to be the vacuum state $|0,0\rangle$, the short term solution is

$$
|\rho(t)\rangle = exp\left[-igt\left(\mathcal{F} + \mathcal{F}^{\dagger} - (\tilde{\mathcal{F}}^{\dagger} + \tilde{\mathcal{F}})\right)\right]|0,0\rangle.
$$

This is the exact solution of the master equation

Conti....

- Now $\mathcal{F}$ and $\mathcal{F}^{\dagger}$ satisfy $SU(1,1)$ Using the disentanglement theorem

$$|\rho(t)\rangle \propto exp\left[-\Gamma\mathcal{F}^{\dagger}\right] exp\left[-\Gamma\tilde{\mathcal{F}}^{\dagger}\right]|0,0\rangle$$

which is an eigenstate of $\mathcal{G}$ and $\tilde{\mathcal{G}}$ with the eigenvalues Γ, given by

$$\Gamma = \frac{2igtsinh\phi}{2\phi cosh\phi - igtsinh\phi},$$

where $\phi^2 = \frac{3(gt)^2}{4}$.

- In usual notation $|\rho\rangle \equiv \rho = |\psi\rangle\langle\psi|$ and $|0,0\rangle \equiv |0\rangle\langle 0|$, then ψ reads as

$$|\psi\rangle \propto e^{[-\Gamma\mathcal{F}^{\dagger}]}|0\rangle,\ \langle\psi| \propto \langle 0|e^{[-\Gamma\mathcal{F}]}.$$

Conti....

- The steady state solution, we rewrite master equation

$$\frac{\partial}{\partial t}|\rho\rangle = \left[(\mathcal{F}-\tilde{\mathcal{F}}^{\dagger})(-ig+\frac{\kappa}{2}\tilde{\mathcal{F}})-(\tilde{\mathcal{F}}-\mathcal{F}^{\dagger})(ig+\frac{\kappa}{2}\mathcal{F})\right]|\rho\rangle.$$

- The steady state solution is an eigenvalue equation

$$\tilde{\mathcal{F}}|\rho\rangle = \frac{2ig}{\kappa}|\rho\rangle,$$
$$\mathcal{F}|\rho\rangle = -\frac{2ig}{\kappa}|\rho\rangle.$$

- In the usual notation for ψ it is given as

$$\mathcal{F}|\psi\rangle = -\frac{2ig}{\kappa}|\psi\rangle,\ \langle\psi|\mathcal{F}^{\dagger} = \langle\psi|\frac{2ig}{\kappa}.$$

Examples I

If $\mathcal{F} = ab$ and $\tilde{\mathcal{F}} = \tilde{a}\tilde{b}$, then the short term solution is the Caves-Schumaker state

$$|\rho(t)\rangle \propto exp\left[-\Gamma a^\dagger b^\dagger\right] exp\left[-\Gamma \tilde{a}^\dagger \tilde{b}^\dagger\right] |0,0\rangle$$

and the steady state is the pair coherent state

$$\tilde{a}\tilde{b}|\rho\rangle = \frac{2ig}{\kappa}|\rho\rangle,$$
$$ab|\rho\rangle = -\frac{2ig}{\kappa}|\rho\rangle.$$

Examples II

If $\mathcal{F} = a^2$, and $\tilde{\mathcal{F}} = \tilde{a}^2$, then the short term is the Yuen state

$$|\rho(t)\rangle \propto exp\left[-\Gamma a^{\dagger 2}\right] exp\left[-\Gamma \tilde{a}^{\dagger 2}\right]|0,0\rangle$$

and steady state is the Cat state

$$\tilde{a}^2|\rho\rangle = \frac{2ig}{\kappa}|\rho\rangle,$$
$$a^2|\rho\rangle = -\frac{2ig}{\kappa}|\rho\rangle.$$

Examples III

- Two boson creation and annihilation operators
- $\mathcal{F} = a + \beta a^{\dagger 2}$ and $\tilde{\mathcal{F}} = \tilde{a} + \beta \tilde{a}^{\dagger 2}$
- the canonical conjugates $\mathcal{G}_0^\dagger = \frac{a^{\dagger 2}}{2}\frac{1}{n_a+1}$, $\mathcal{G}_1^\dagger = \frac{a^{\dagger 2}}{2}\frac{1}{n_a+2}$, $\tilde{\mathcal{G}}_0^\dagger = \frac{\tilde{a}^{\dagger 2}}{2}\frac{1}{\tilde{n}_a+1}$ and $\tilde{\mathcal{G}}_1^\dagger = \frac{\tilde{a}^{\dagger 2}}{2}\frac{1}{\tilde{n}_a+2}$.
- Here $n_a = a^\dagger a$ and $\tilde{n}_a = \tilde{a}^\dagger \tilde{a}$, satisfies $[\mathcal{F}, \mathcal{G}_i^\dagger] = 1$ and $[\tilde{\mathcal{F}}, \tilde{\mathcal{G}}_i^\dagger] = 1$.
- The short term solution

$$|\rho(t)\rangle \propto exp\left[-\Gamma(a^\dagger + \beta a^2)\right] exp\left[-\Gamma(\tilde{a}^\dagger + \beta \tilde{a}^2)\right] |0,0\rangle,$$

- The steady state:

$$\begin{aligned} (a + \beta a^{\dagger 2})|\rho\rangle &= \frac{2ig}{\kappa}|\rho\rangle, \\ (\tilde{a} + \beta \tilde{a}^{\dagger 2})|\rho\rangle &= 0 - \frac{2ig}{\kappa}|\rho\rangle. \end{aligned}$$

Examples IV

- If $\mathcal{F} = ab + \beta a^\dagger b^\dagger$ and $\tilde{\mathcal{F}} = \tilde{a}\tilde{b} + \beta\tilde{a}^\dagger\tilde{b}^\dagger$
- The canonical conjugates $\mathcal{G}_0^\dagger = \frac{a^\dagger b^\dagger}{2}\frac{1}{n_b+1}$, $\mathcal{G}_1^\dagger = \frac{a^\dagger b^\dagger}{2}\frac{1}{n_a+1}$, $\tilde{\mathcal{G}}_0^\dagger = \frac{\tilde{a}^\dagger\tilde{b}^\dagger}{2}\frac{1}{\tilde{n}_b+1}$ and $\tilde{\mathcal{G}}_1^\dagger = \frac{\tilde{a}^\dagger\tilde{b}^\dagger}{2}\frac{1}{\tilde{n}_a+1}$.
- Here $n_a = a^\dagger a$, $n_b = b^\dagger b$, $\tilde{n}_a = \tilde{a}^\dagger\tilde{a}$ and $\tilde{n}_b = \tilde{b}^\dagger\tilde{b}$, satisfies $[\mathcal{F}, \mathcal{G}_i^\dagger] = 1$ and $[\tilde{\mathcal{F}}, \tilde{\mathcal{G}}_i^\dagger] = 1$.
- The short term solution

$$|\rho(t)\rangle \propto exp\left[-\Gamma(a^\dagger b^\dagger + \beta ab)\right] exp\left[-\Gamma(\tilde{a}^\dagger\tilde{b}^\dagger + \beta\tilde{a}\tilde{b})\right]|0,0\rangle,$$

- The steady state :

$$\begin{aligned}(ab + \beta a^\dagger b^\dagger)|\rho\rangle &= \frac{2ig}{\kappa}|\rho\rangle,\\ (\tilde{a}\tilde{b} + \beta\tilde{a}^\dagger\tilde{b}^\dagger)|\rho\rangle &= -\frac{2ig}{\kappa}|\rho\rangle.\end{aligned}$$

Conclusion

- We have constructed a general master equation for Janus Faced commutation relations $[\mathcal{F}, \mathcal{G}^{\dagger}] = 1$ and shown that the eigenstate of $\mathcal{G}$ occurs as a short term solution and the eigenstate of $\mathcal{F}$ occurs as a steady state solution.
- We have also shown that the master equation works for all the Janus Faced coherent state operators.
- We have also shown that the short term solution is an exact solution for the master equation and the long term solution is an eigenvalue equation.
- Most of the coherent states discussed in this paper are entangled states such as pair coherent state, the cat state etc and we expect that this master equation will throw light on production in the laboratory.