Assignment 1 Math 324 (Test on Jan. 30)

1. Let E_i's denote measurable sets. Prove the following statements.

(a) Let E_1 and E_2 be two measurable subsets of $\mathbb{R}$. Prove that

$$m(E_1 \bigcup E_2) + m(E_1 \bigcap E_2) = m(E_1) + m(E_2).$$

(b) If $E_1 \subseteq E_2 \subseteq E_3 \cdots$, then

$$m(\bigcup_i E_i) = \lim_{i \to \infty} m(E_i).$$

(c) If $E_1 \supseteq E_2 \cdots$ and $m(E_1) < \infty$, then

$$m(\bigcap_i E_i) = \lim_{i \to \infty} m(E_i).$$

2. Let $E \subseteq \mathbb{R}$. Prove that the following statements are equivalent:

(a) E is measurable.

(b) For each $\epsilon > 0$, we can determine an open set $G \subseteq \mathbb{R}$ such that $E \subseteq G$ and $m^*(G - E) < \epsilon$.

(c) There exists a subset O of $\mathbb{R}$ such that O is an intersection of countably many open sets, $E \subseteq O$ and $m^*(O - E) = 0$.

3. Let E be a set with $m^*(E) < \infty$. Prove that E is measurable if and only if for every $\epsilon > 0$, there is a finite union of B of open intervals such that $m^*(E \bigtriangleup B) < \epsilon$, where $E \bigtriangleup B$ denotes $(E - B) \bigcup (B - E)$.