# Circles to Numbers

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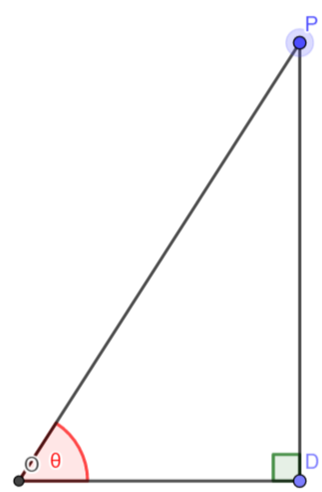
When learning Calculus or Analysis for the first time, it can feel as if the entire subject is only about computation; one can forget the forest for the trees so to speak, and it all seems a bit arbitrary. However, there is no subject in mathematics that is an island; and Analysis is one such branch which is at the heart of nearly all branches of mathematics. For this particular article, I want to show how basic geometry that one learns before $11^{th}/12^{th}$ is connected with the abstract analysis that comes afterwards.(The inspiration comes from [4])

## Geometric Point of View

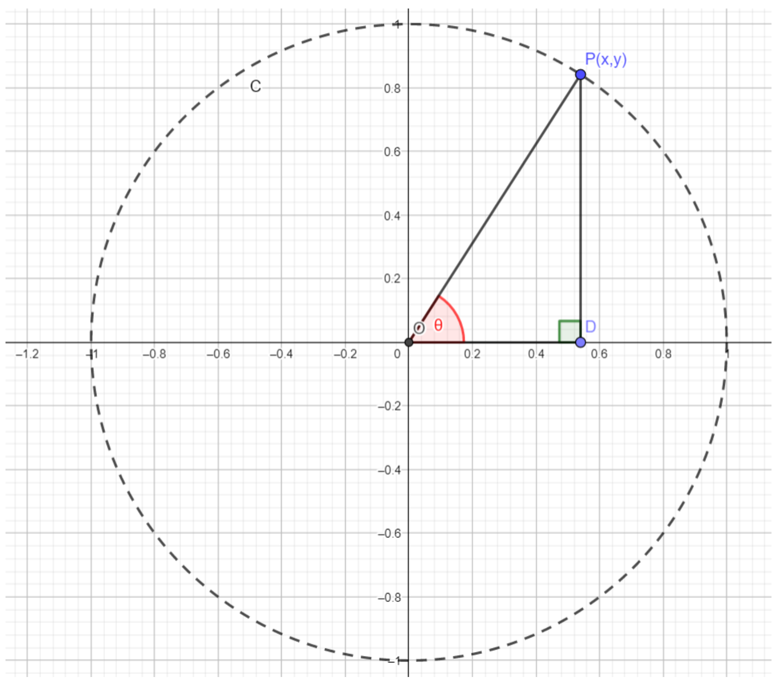
In school, one usually learns about trigonometry from triangles and circles. Just to recall, if one has a right angled triangle $\Delta PDO$, then the sine and cosine of $\angle POD = \theta$ is given by:



Classical Definitions

$$\sin(\theta) = \frac{PD}{OP}$$
$$\cos(\theta) = \frac{OD}{OP}$$

These are the fundamental trigonometric definitions from which everything else in the subject can be derived. However, this definition can only work for angles which are less than 90°- we can consider them to be functions mapping $(0, 90°)$ radians to $\mathbb{R}$- so what do we do for more general triangles, or in mathematical terms, how do we extend the *domain* of definition for other angles? Here's one such example of the extension of the domain $(0, 90°)$ :




Extension of domain to $[0, 360°)$

Consider a point $P(x, y)$ subtending an angle of $\theta$ on the unit circle. Then we have:

$$x = \cos(\theta)$$
$$y = \sin(\theta)$$

In fact a different system of measuring angles on circles is adopted henceforth to allow greatest simplicity : 1 radian is the angle subtended by an arc of length 1 on the unit circle.

Hence in this system for an arc length $l$, the angle it subtends is:

$$\theta = \frac{l}{1}$$

Hence, the new domain of definition is: $[0, 2\pi)$. In this system, you can clearly see that both the $\sin, \cos$ functions are bounded by 1(exercise: try applying the Pythagorus theorem and play with the algebraic expression). Also you can prove the following addition theorems for both sin and cos functions by simple geometry:

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$
$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

# Analytic Point of View

When one starts learning analysis or calculus, they are usually told to forget about the old Euclidean viewpoint. "Look here", they say; "We have to be rigorous- we can't rely on these pictures for a completely self contained definition! We must have an independent way of defining the trigonometric functions that only depends on the properties of real numbers..."

Flipping through a textbook, one would usually come across a definition like the following:

$$\sin(\theta) = \frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots + (-1)^{n+1}\frac{\theta^{2n+1}}{(2n+1)!} + \ldots, \forall \theta \in \mathbb{R}$$
$$\cos(\theta) = \frac{1}{0!} - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots + (-1)^{n}\frac{\theta^{2n}}{(2n)!} + \ldots, \forall \theta \in \mathbb{R}$$

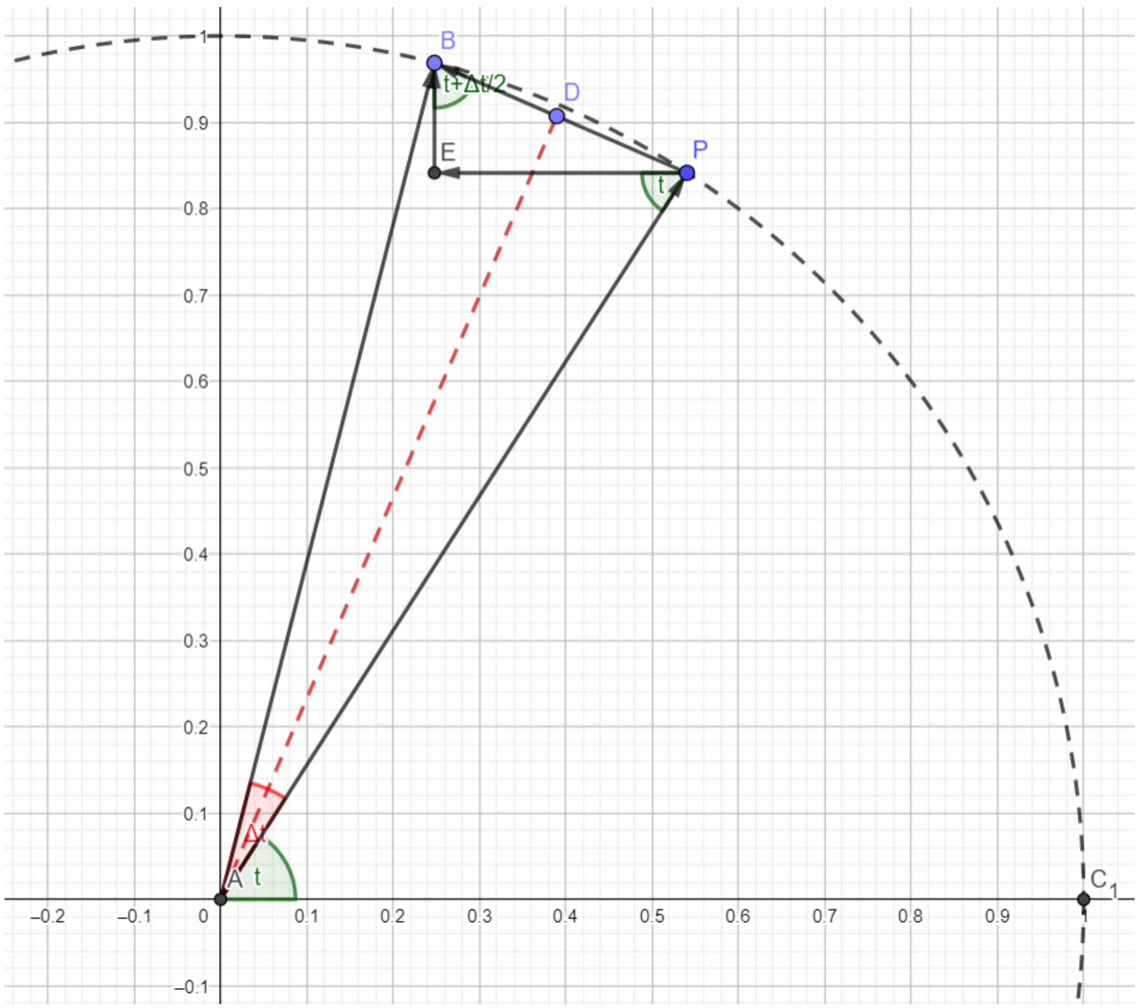
Leaving aside the meaning of such infinite sums for the moment, at first sight, this seems totally different from the definition you saw just before. The earlier definition you saw is completely geometric, while this definition is completely algebraic. How can there be any connection between the two?

What we really need is a way to characterize $\sin : [0, 2\pi) \to \mathbb{R}$ and $\cos : [0, 2\pi) \to \mathbb{R}$ which are from the euclidean point of view in terms completely in analysis. A preliminary idea might be to get a relation between functions that is independent of the pictures used to get there. One relation which these functions:

$$x(\theta) = \cos(\theta)$$
$$y(\theta) = \sin(\theta)$$

already satisfy $x^2(\theta) + y^2(\theta) = 1$. But that is not very useful in the direct sense, as there is no direct path to the analytic definitions. So what do we do now?

It has been found again and again that when dealing with functions/general object which return numbers, it is sometimes easier to describe 'changes' in quantities rather than the actual functions themselves. In a sense, it boils down to asking what happens to the function when I slightly tweak the input? How it responds to small changes of input(differentiation) can be used to characterize the entire function by repeatedly 'adding together' these small changes(integration)- this is the essence of Calculus. Let us see what happens when we ask the same question here;(note we use $t$ as the parameter for the angle for an analogy with time)




From the figure(as an exercise in high school geometry) and the algebraic addition formulae we can see the following:

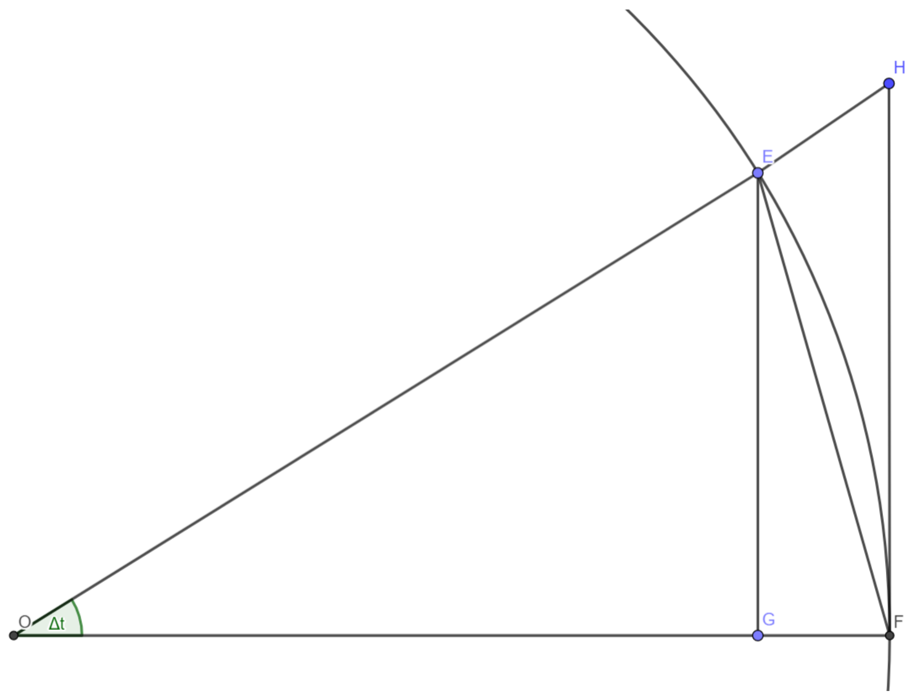
$$\frac{\Delta x}{\Delta t} = \frac{x(t+\Delta t) - x(t)}{\Delta t} = -\sin\left(t + \frac{\Delta t}{2}\right)\left(\frac{\sin\left(\frac{\Delta t}{2}\right)}{\frac{\Delta t}{2}}\right)$$

$$\frac{\Delta y}{\Delta t} = \frac{y(t+\Delta t) - y(t)}{\Delta t} = \cos\left(t + \frac{\Delta t}{2}\right)\left(\frac{\sin\left(\frac{\Delta t}{2}\right)}{\frac{\Delta t}{2}}\right)$$

To proceed further, we need a lemma[1]:

Lemma

$$\lim_{\Delta t \to 0^+} \sin(\Delta t) = 0$$
$$\lim_{\Delta t \to 0^+} \cos(\Delta t) = 1$$
$$\lim_{\Delta t \to 0} \frac{\sin(\Delta t)}{\Delta t} = 1$$




The first two theorems can be proved by noting that both $\sin, \cos$ are monotonically increasing and decreasing respectively for small $\Delta t > 0$, and thus they are invertible. Hence for example, if someone gives an $\epsilon > 0$, we can find a suitable $\Delta t_\epsilon$ such that

$$0 < \sin(\Delta t) < \epsilon, 0 < \Delta t < \Delta t_\epsilon = \sin^{-1}(\epsilon)$$

Hence the limit exists for the first equation, and the second equation can be proved by noting $\cos(\Delta t) = \sqrt{1 - \sin^2(\Delta t)}$.

By considering the figure inscribed in a unit circle above, we can see that:

$$\text{Area}(\Delta EFO) \leq \text{Area}(\text{Arc}EFO) \leq \text{Area}(\Delta HFO)$$

Hence we get:

$$\frac{1}{2}\sin(\Delta t) \le \frac{\Delta t}{2\pi} * \pi \le \frac{1}{2}\frac{\sin(\Delta t)}{\cos(\Delta t)}$$

Dividing by $\frac{\sin(\Delta t)}{2}$ on both sides:

$$\begin{aligned} 1 &\le \frac{\Delta t}{\sin(\Delta t)} \le \frac{1}{\cos(\Delta t)} \\ \implies \cos(\Delta t) &\le \frac{\sin(\Delta t)}{\Delta t} \le 1 \end{aligned}$$

Taking the limit $\Delta t \to 0^+$, we get the last relation by Sandwich Theorem. If $\Delta t$ is negative, then plug in $\Delta u = -\Delta t$ and get the same relation.

Using the addition formulae and taking the limit $\Delta t \to 0$ on both sides we obtain the fact that both $x, y$ are differentiable functions, and we get the following system of equations:

$$\begin{aligned} \frac{dx}{dt} &= -y(t) \\ \frac{dy}{dt} &= x(t) \end{aligned}$$

along with the initial conditions $x(0) = 1, y(0) = 0$.

Aha! After a long struggle, we have reduced the geometric picture down to relations between purely functions- now Analysis can come in naturally!

We already know that the sine and cosine functions are characterized by these equations. But how do we obtain a purely algebraic expression for them from these? If we consider $\theta(t) = t \implies \theta'(t) = 1$, as in the $\theta$ variable is a function of time and the angular speed is 1, then we have 'equations of motion' for a particle in analogy with physics.

$$\begin{aligned} \frac{dx}{dt} &= -y(t) \\ \frac{dy}{dt} &= x(t) \end{aligned}$$

Now this looks a bit clumsy, but by writing everything in a vector form we can greatly compact things; just remember that for a vector function of a real number, we take limits, derivatives and integrals component-wise.

$$\begin{aligned} \lim_{n\to\infty} \mathbf{r}_n &= \left(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n\right) \\ \frac{d\mathbf{r}}{dt} &= \left(\frac{dx}{dt}, \frac{dy}{dt}\right) \\ \int_a^b \mathbf{r}(t)dt &= \left(\int_a^b x(t)dt, \int_a^b y(t)dt\right) \end{aligned}$$

whenever the quantities on the right hand side exist. In vector form, if we take $\mathbf{r}(t) = (x(t), y(t))$ as the position vector, the actual trajectory of the particle is:

$$\frac{d\mathbf{r}}{dt} = \mathbf{f}(\mathbf{r}(t))$$

where $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as:

$$\mathbf{f}(x, y) = (-y, x), \forall x, y \in \mathbb{R}$$

You can interpret this equation as giving the velocity of the particle for each position and time. Now if you know the velocity at one point of time, you roughly know where the particle is going for the next small time. At the next position, you plug that position back into the equation and find out the new velocity, and from that where you go next and so on ad infinitum...

(Obviously the idea only works if the functions in our differential equations are 'nice' enough in a sense. For more particulars see [2],[3])

As a small exercise, try to compute the dot product of $\mathbf{r}$ with $d\mathbf{r}/dt$... you will see that the velocity of the particle is perpendicular to the position at each point of time, and moreover the magnitude of this velocity is fixed; this makes sense as we derived these equations from a particle travelling along a circle with uniform angular speed; so the instantaneous velocity must be perpendicular to the position.

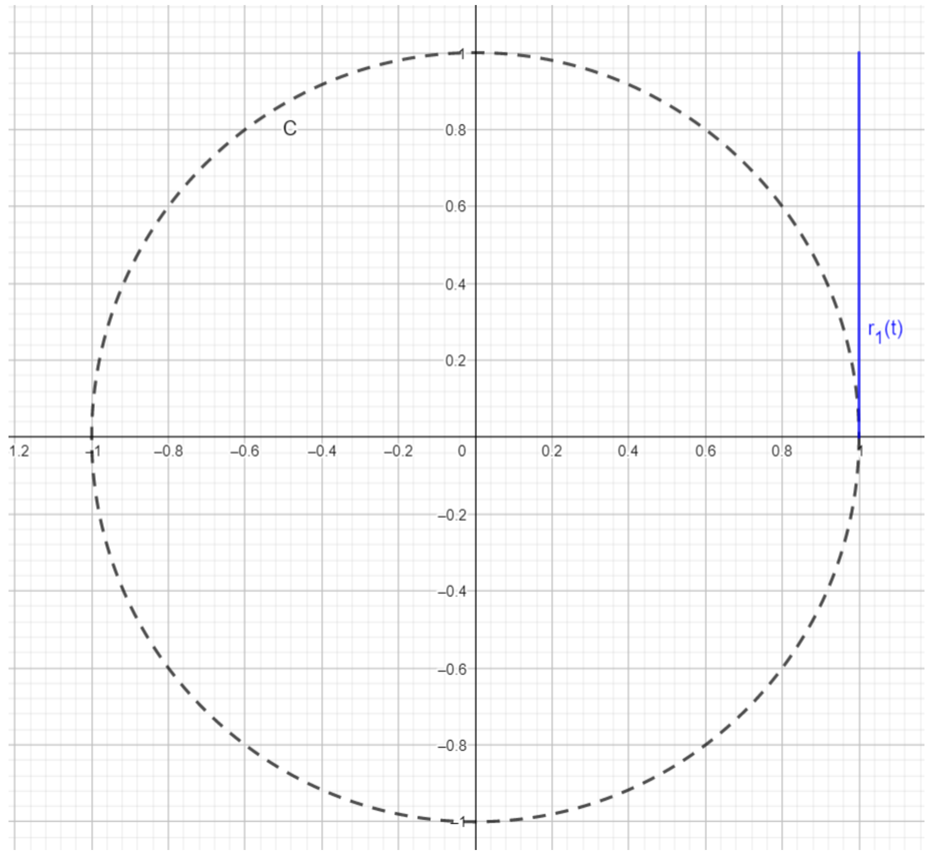
All this still doesn't give us the analytic forms that we started our quest for. However, let us go on with the rough intuitive idea developed above, and let us consider a particle starting at the point $(1, 0)$:
Then the velocity at this initial point is given as:

$$\frac{d\mathbf{r}}{dt}(0) = (0, 1)$$

This means that it tends to go upwards with a unit speed initially. As we discussed earlier, let us consider as if the particle is actually going in the upward direction for some amount of time; what is the position of the particle after some time? By the fundamental theorem of calculus(applied component wise):

$$\mathbf{r}_1(t) - \mathbf{r}_1(0) = \int_0^t (0, 1)dt = (0, t) \implies \mathbf{r}_1(t) = (1, t)$$

The trajectory is plotted below:



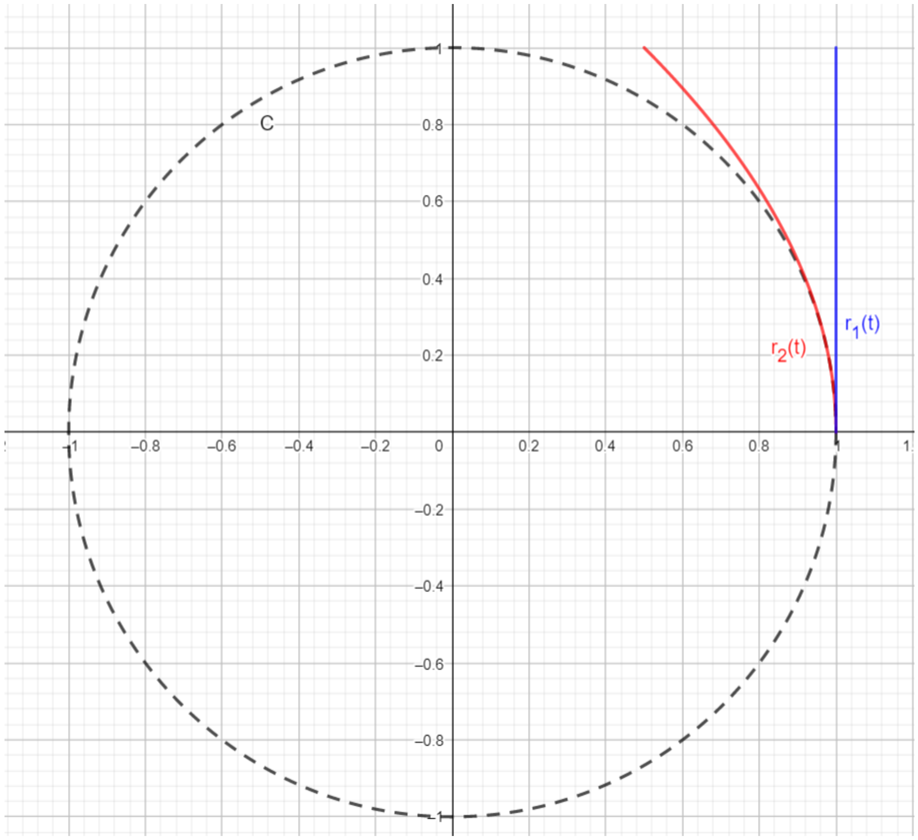

As you can see, this isn't quite right; for initial time its a good enough approximation, but it doesn't quite curve the way its supposed to. What do we do? Well, we can think of the following: The particle travels some distance in the direction of this trajectory, but the velocity is still given by the original differential equation; in effect we have the following modification:

$$\frac{d\mathbf{r}_2}{dt} = (-t, 1)$$

for every single instant; as that gives what the next step should be after time $t$; and you can make that initial time step as small or as big as you please, so we get the vector for velocity for each point of time for the 'next' instant of time. This velocity vector points in a small direction towards the left after some time passes, so the new trajectory should curve a bit; again using the fundamental theorem of calculus we get:

$$\mathbf{r}_2(t) - \mathbf{r}_2(0) = \int_0^t (-t, 1)dt = \left(-\frac{t^2}{2!}, t\right) \implies \mathbf{r}_2(t) = \left(1 - \frac{t^2}{2!}, t\right)$$

The trajectory looks like(for a small time):




Now you have to admit this looks exciting! Just by the idea of approximating the real trajectory of the particle we are getting a sequence of trajectories which are curving the correct way the more number of times we iterate the procedure, and the *expressions for the approximations look exactly like the analytic definitons*. To actually prove that analytic definitions hold, we must in essence prove the following:

Finale?

$$\mathbf{r}(t) = \lim_{n\to\infty} \mathbf{r}_n(t) \text{ is the unique solution of the differential equation}, \forall t \in [0, 2\pi)$$

And wouldn't you know it, but that is exactly the content of the famous *Picard-Lindelof Theorem*[2], which states that under certain conditions on $\mathbf{f}$, the differential equation we started with has a unique solution atleast on some small interval. Hence the disparate definitions of $\sin, \cos$ actually are connected to each other through the medium of differential equations!- and that is just the beginning! Differential equations connect other disparate areas and can be used as a way to define special functions of analysis in the most natural way...*so we too can see further from the shoulders of Differential equations...*

## Appendix(Proofs of convergence)

Consider the series for $x_n(t), t \in [0, 2\pi)$. For $n > m$ consider:

$$
\begin{aligned}
|x_n(t) - x_m(t)| &\leq \left| \sum_{k=m+1}^{n} (-1)^{k+1} \frac{t^{2k+1}}{(2k+1)!} \right| \\
&\leq \sum_{k=m+1}^{n} \frac{|t|^{2k+1}}{(2k+1)!} \\
&\leq \sum_{k=m+1}^{n} \frac{(2\pi)^{2k+1}}{(2k+1)!} \\
&\leq \sum_{k=2m+1}^{2n+1} \frac{(2\pi)^k}{k!} b \leq \frac{(2\pi)^{2m+1}}{(2m+1)!} \sum_{k=0}^{2n-2m} \frac{(2\pi)^k}{(2m+1)^k}
\end{aligned}
$$

Now for $m > 6$, $2m+1 > 2\pi$ and $2\pi/(2m+1) < 1/2$ Let $c = \frac{2\pi}{7}$. Hence we have:

$$
|x_n(t) - x_m(t)| \leq \frac{(2\pi)^{2m+1}}{(2m+1)!} \sum_{k=0}^{2n-2m} \frac{(2\pi)^k}{(2m+1)^k} \leq \frac{(2\pi)^{2m+1}}{(2m+1)!} \sum_{k=0}^{\infty} \frac{(2\pi)^k}{(2m+1)^k} \leq \frac{(2\pi)^{2m+1}}{(2m+1)!} \frac{1}{1 - \frac{2\pi}{2m+1}} \leq Ac^m
$$

where $A \in \mathbb{R}_{\geq 0}$ is a constant. Hence for any given $\epsilon > 0$, we can always find an $N$ such that for any $n, m > N$, we have:

$$
|x_n(t) - x_m(t)| < \epsilon
$$

Note that by the algebra we did previously, this $\mathbb{N}$ is the same for any $t \in [0, 2\pi)$; hence $x_n : [0, 2\pi) \to \mathbb{R}$ is a *Uniformly Cauchy Sequence of Functions*. We can write a very similar proof for $y_n$. Moreover, since we have:

$$
\begin{aligned}
\frac{dx_{n+1}}{dt}(t) &= -y_n(t) \\
\frac{y_{n+1}}{dt}(t) &= x_n(t)
\end{aligned}
$$

it is possible to show that $x_n, y_n$ and their derivatives all are uniformly converging on $[0, 2\pi)$. Referring to Theorem 7.17 from Rudin's book[3], we have:

$$
\begin{aligned}
\frac{dx}{dt}(t) &= \lim_{n\to\infty} \frac{dx_{n+1}}{dt}(t) = \lim_{n\to\infty} -y_n(t) = -y(t) \\
\frac{dy}{dt}(t) &= \lim_{n\to\infty} \frac{dy_{n+1}}{dt}(t) = \lim_{n\to\infty} x_n(t) = x(t)
\end{aligned}
$$

Hence we have atleast one solution of the system of differential equations. Now suppose we have two distinct solutions of this differential equations, $(x(t), y(t))$ and $(x_1(t), y_1(t))$. Consider the function [5]:

$$
h(t) = (x(t) - x_1(t))^2 + (y(t) - y_1(t))^2
$$

Then $h$ is differentiable and we have:

$$
h'(t) = -2(x(t) - x_1(t))(y(t) - y_1(t)) + 2(x(t) - x_1(t))(y(t) - y_1(t)) = 0
$$

Hence $h$ is constant over $[0, 2\pi)$. Since the initial conditions are same, $h(0) = 0$, which implies that the solution of the differential equations is unique over $[0, 2\pi)$. It is in this sense we say that we get back the analytic definitions of $\sin, \cos$ from the geometric ones, since we made the differential equations from the geometric $\sin, \cos$.

# References

[1] *What is Mathematics?* by Courant and Robbins

[2] *Differential Equations with Applications and Historical Notes* by Simmons

[3] *Principles of Mathematical Analysis* by Walter Rudin

[4] *Visual Complex Analysis* by Tristan Needham

[5] *Calculus-I* by Apostol