Cyclones, Milkweeds, and Hedgehogs

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Abstract

The article explores a theorem that states that even dimensional spheres do not admit a smooth non-vanishing nowhere zero tangent vector field. The 3 dimensional analogue of this is called the Hairy ball theorem, or as the Europeans say, the Hedgehog theorem, in layman's term which says that if you have a ball with hair sprouting out of every point, it cannot be combed without creating a bald spot or a cow-lick!

"Once the storm is over, you won't
remember how you made it through, how
you managed to survive.You won't even be sure in fact, whether the storm is really over.
But one thing is certain, when you come out of the storm you won't be the same person who walked in.
That's what this storm is all about."
Haruki Murakami

It ain't no matter if the stuff introduced in sections 1, 2, and 3 are unfamiliar to you, reader. You can enjoy the Hairy consequences just as much if you start right from section 4. Happy read!

1 Vector fields, sections, Tangent and vector bundles

Tangent bundle: Let M be s smooth manifold, and let for any point x ∈ M, T_xM typify the Tangent space anchored at x. Then the tangent bundle TM of M is defined as follows:
TM := $\cup_{x\in M}\{x\} \times T_xM$

Vector bundle: A vector bundle is a triple (π, E, B), where E and B are smooth manifolds, and π E ⟶ B is a smooth map satisfying:
$(\star)\pi$ is surjective.
$(\star\star)$There is an open cover (U_i) i ∈ I, of B with a collection of diffeomorphisms h_i $\forall i$ where
$h_i : \pi^{-1}(U_i) \longrightarrow U_i \times R^n$
$h_i(\pi^{-1}(x)) = x \times R^n$
$(\star\star\star)\forall i, j \in I$ the map $h_i(h_j)^{-1}$ restricted to $(U_i \cap U_j) \times R^n$ is smooth.

Section: A (smooth) section of a vector bundle (π, E, B) is a (smooth)map m:$B \longrightarrow E$ such that $\pi \circ m = id_B$

Vector field: If M is a smooth manifold, a (smooth) vector field F on M is a section F:M⟶TM.
For our purpose, a vector field on S^{n-1} is a section of the tangent bundle.
In a more prosaic but visually vibrant term, it is a map F:$S^{n-1} \longrightarrow R^n$ such that $\forall p \in S^2$ we have $<F(p),p>=0$, F is continuous, where, $<,>$ can be taken as the standard dot product.

2 The Mother of Hedgehogs and Hairy balls!

Theorem 2.1. *The sphere S^{n-1} admits an everywhere non-zero smooth vector field iff n is even.*

Proof. (Necessary) Assume that n is even, let n=2m.
Then we can think of S^{n-1} as the space of all unit vectors in $\mathbf{R}^{2m} \cong \mathbf{C}^m$.

To picture this, mentally set $m = 1$.
Define a map v:$S^{n-1} \longrightarrow C^m$
x $\mapsto$ ix.
Clearly $\forall$ x $\in S^{n-1}$, $\langle x, v(x)\rangle = 0$ and $\parallel v(x) \parallel= 1$.
$\therefore$ v is a well defined non-zero non vanishing smooth vector field on S^{n-1}. So we have that if n is even, a non vanishing smooth non-zero vector field is admitted for S^{n-1}

(Sufficient)
We argue using contrapositivity.
Let n is odd and let v be the map defined as above. Then such a map v would give us a homotopy between the identity and the antipodal map A:$S^{n-1} \longrightarrow S^{n-1}$ defined as usual A(x)= -x.
In particular, the specific homotopy is:
H: [0,1] $\times$ $S^{n-1} \longrightarrow S^{n-1}$
H(t,x)= $\cos{(\pi t)}x$ + $sin(\pi t)v(x)$.
It's is easy to see that the map is indeed a homotopy.
But this gives us a contradiction, because homotopic functions have equal degrees, but since we know that the degree of the identity map is 1 and the degree of the antipodal map is -1,
we get a contradiction. □

3 The Hairy Ball theorem

Theorem 3.1. *There does not exist an non-zero vector field on a 2-sphere* S^2

Proof. Just set n = 3 in the above theorem. □

4 The theorem spelled out.

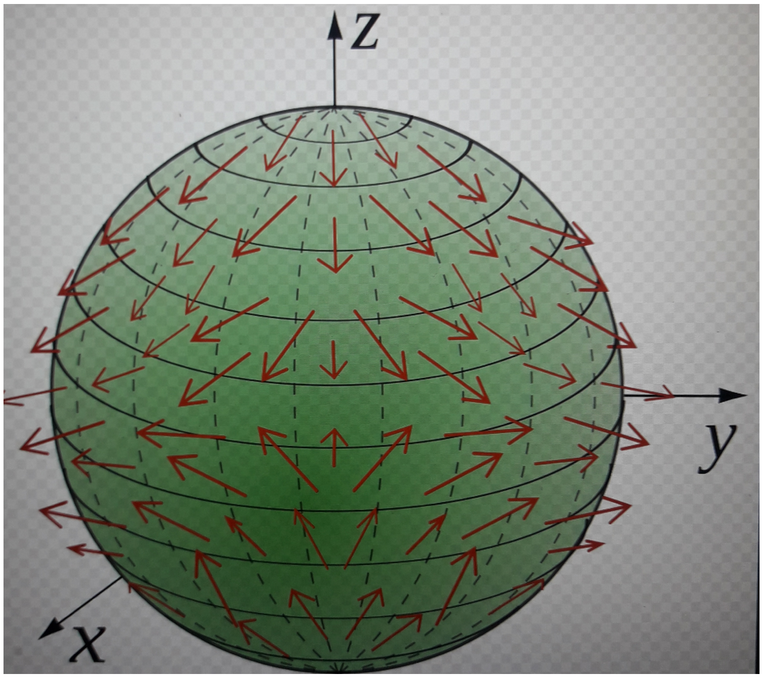

Figure 1: Vector field on a 2-sphere.

Take a ball S^2 or anything homeomorphic to it with hairs sprouting out of it's surface from every point, now if you try to smoothly comb the hair to make them lie flat, you'll always end up either with a cowlick(At least one hair standing up) or a bald spot!

Homeomorphism: In simple terms, two things A and B are homeomorphic if you can deform A into B just by stretching, twisting, bending, and wrinkling, but not by ripping, puncturing,shredding, glueing, or pulverizing.

5 Head and Hairy Balls:

Figure 2: A Heady hypnotic Whorl!

Close your mouth, shut all other orifices in your head so that it becomes a genus 0 surface, and thus, homeomorphic to S^2. Now, your head (Baldies, stop reading!) is nothing but a hairy ball, so the above theorem applies. Have you ever wondered why everyone, every single people you met in your life has that swirl at the top of their head or unruly cowlicks all over? If yes, no need to wonder anymore, it's due to a spell casted by the Hairy ball theorem, and there is no escape!

6 Mirrors and Hairy balls:

When the ideal spherical mirror, homeomorphic to a sphere, is illuminated at every point, the "Hairy ball theorem" prescribes the existence of at least one point at which the incident light will be normally reflected. For the more general case of the surface, topologically equivalent to a sphere, which is both reflecting and refracting the "hairy ball theorem" predicts the existence of at least one point, at which the incident light will be normally reflected and also normally refracted.

7 Cyclones and Hairy Balls

One of the most widely used applications of the hairy ball theorem is connected to the Earth's atmosphere. One can look the wind as a vector that is defined continuously everywhere on the surface of the Earth. Then if there is a place on the surface of the Earth where the wind blows, the hairy ball theorem states that there is another point on the Earth's surface where there is no wind. In the physical sense, such point can be considered to be an eye of a cyclone or an anticyclone. Mind-blowing, right? Some readers might object the wind blows in three dimensions and not necessarily just two and hence the theorem is not valid in this case. Well, that is most certainly true, however, in comparison to the diameter of the Earth the height of the atmosphere is negligible and so two-dimensional approximation can be valid. Or one might think of the atmosphere as a set of layers of different heights, in which case the theorem can then be applied to the layers individually.

Ever watched the satellite image of the Earth, if so I'm sure you'd have noticed those white swirls near the poles And yes, that's because of the Hairy ball theorem too!

8 Hedgehogs and Hairy balls

I once met a very friendly hedgehog. He was a fun guy, one with whom you can hang around with in the park in the dark, and have a nice post dinner tête-á-tête with over a glass of beer. Here he is:

Figure 3: My friend!

So one day, having read this theorem, I asked him if he'd allow me to experiment a bit with him. Now the fun guy as he was, he pricked all up with excitement(pun intended)and acquiesced. I asked him to shut his mouth up and I plugged up his ears with mud cakes so that he had genus 0 and I can apply the homeomorphism to make him into a ball. He did and I fed him into the map, and this came out:

Figure 4: My friend morphed into S^2

I studied this hedge ball and indeed there was a bald spot, and also a few cowlicks! Success! I just verified the Hairy ball theorem for a particular case! It was time to get my friend back to normal. Homeomorphisms being invertible, I fed this stupid ball hedge to the other end of the map, the machine chugged for a while and plop! There popped my friend, bright eyed and bushy tailed! He later confessed that the process was intense, and having lost a few spikes, he is now a bit pissed off.

Figure 5: Argg!

9 Moral

The takeaway is, at every moment there is either a storm brewing or placid weather prevailing. Same with life, it's pliable, and thus through every storm you push through, there is calm weather awaiting, and as soon as you take this calmness from granted, somewhere...a a storm is brewing...
Life is a road, a boulevard of shattered shadows and sunlight. It's these thorns and flowers that make life this beautiful...

Life is a song
of time gone long,
of rust on clocks,
of Name on rocks.
of tittering stars on crystal nights,
of mornings cold in callow light,
Of roads we lost
of tears it cost.
To wince and try despite pain
Seekin' and seekin' some meaning in vain
To sing to dance in drumming rain
A toast to blood in thy veins,
winning and losing, life will drain...

10 References

(1)Introduction to Smooth Manifolds: John M.Lee
(2)Algebraic Topolgy:Beckham Mayers lecture notes.
(3)Riemannian Manifolds, an introduction to curvature: John M.Lee.