Going around in circles:

Counting circular permutations with repeated objects is not as easy as it looks!

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So the other day I had a quiz on permutations and combinations. Easy stuff right? We have all heard the formula for the number of permutations of n objects around a circle; (n-1)! of course! Let us first take a look at why this is so.

The number of ways to arrange n objects in a line is clearly n!. The only thing that is different in a circle is that since a circle is cyclic, the starting point does not make any difference. Thus the n! arrangements in a line are split into equivalence classes depending on the starting point, which are of course disjoint. Each arrangement has n equivalent arrangements, depending on the starting point. Thus we have $\frac{n!}{n}$= (n-1)! Different arrangements for objects arranged in a circle.

And permutations where objects are repeated? No biggie, it's $\frac{n!}{n1!*n2!...nk!}$where n_1, n_2 are number of objects of the first kind, second kind, etc because we have n! arrangements in the line, out of which n_1 objects of the first kind can be permuted in n_1! ways which would still be the same arrangement, n_2 objects of the second kind can be permuted in n_2! ways, and so on.

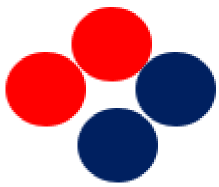
What if we combined the 2 problems, permutations around a circle with repeated objects? Still doesn't seem like a difficult proposition, does it? One would logically think it to be $\frac{(n-1)!}{n1!*n2!...nk!}$ especially if we look at the logic behind both the formulae. (n-1)! ways to arrange around a circle, out of which n_i similar objects can be permuted in n_i! ways, seems intuitively correct.

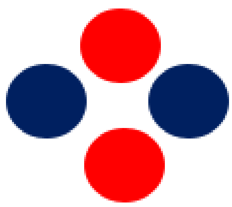
Turns out…it's not. Or more accurately, not always. So in the quiz, I had been asked to find the number of ways to arrange 4 red and 5 blue (identical) balls around a circle, immediately applied the above logic to get $\frac{8!}{5!*4!}$=14 ways. I ran a program just to double check and was delighted to find I was correct with the 14 ways listed one by one.

….until my friend texted me to discuss answers. He had been given the same question but with the values 4 red and 2 blue balls around the circle (this was a particularly tech savvy instructor who liked showing off his coding skills and used some fancy script to shuffle questions and values for everyone). He applied the same formula, to get $\frac{5!}{4!*2!}$=……2.5 ways?! Not even a whole number?

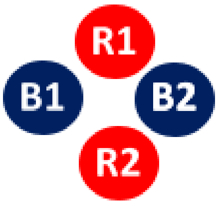
Huh? What is going on here? The answer, as is to many mysterious things in the universe, is symmetry. While coming up with this 'intuitive' formula, we also indulged in the reverse of a common error encountered while solving many combinatorics questions; double discounting instead of double counting.

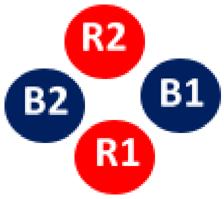
So what really happened here? To understand, let us take the simplest case possible: we have to arrange 2 red and 2 blue balls in a circle. Play around with them and we see that there are only 2 possible ways to arrange them in a circle, i.e. with both balls of the same colour together ('cis') or alternating ('trans').





Why is our formula wrong? Naively applying it gives us $\frac{3!}{2!*2!}$=1.5 which is obviously incorrect. But where exactly are we going wrong? Let us name the balls R1, R2, B1, B2 and try to arrange them in a line. There are clearly 4!=24 ways to do this. When we arrange them in a circle, we get a lower number (3!) because we divide the 24 ways into equivalence classes which are basically the same cycle but only with a different starting point and taking only one representative from each equivalence class. For example, observe the two arrangements below:





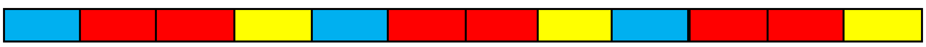
They are both part of the same equivalence class (**even when considering the balls as distinct**) and counted only once in the number of circular arrangements of four balls. Because the second arrangement is just the first arrangement but starting with R2 instead of R1.

Now according to the formula, we again divide the number of circular arrangements into equivalence classes, this time based on the indistinction between balls of the same colour. We divide the result by $n_1!n_2!....n_k!$ (here 2!*2!=4) because each of the n_k elements of similar colour can be permuted in $n_k!$ ways between themselves.

Here our fallacy lies in assuming all the equivalence classes are of the same size and dividing by 4 throughout. For instance the 'cis' case does have four representatives remaining out of the six possible circular arrangements (i.e. R1R2B1B2, R2R1B1B2, R2R1B2B1, R2R1B1B2). We have assumed here that the 'trans' case also has 4 representatives (which should have been R1B1R2B2, R1B2R2B1, R2B2R1B1, R2B1R1B2). But if we notice carefully, the last two cases have already been eliminated in the first step (making the linear arrangements into circular) on account of being the same cycle as the first two but differing just by starting point. By dividing throughout by 4, we are eliminating them twice! The trans arrangement has only two representatives among the six. If we take into account symmetry into our aforementioned formula and divide accordingly, we do get $\frac{4}{4}+\frac{2}{2}$=2 which is the correct answer.

In other words, with symmetric arrangements, by permuting similar elements with each other, we end up with some arrangements that are the same as ones that are eliminated because of having a different starting point and eliminate them again. The reason I got the correct answer in my quiz is because with the numbers I had been given, there was no symmetry!

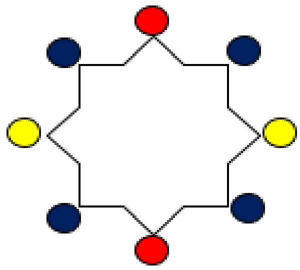
Now that we have realized where we were going wrong, let us now try to get at the correct approach to the problem. Since all the ado is about symmetric arrangements, the first task is to precisely formulate what these are and a way to enumerate them. A symmetric arrangement consisting of a repeating pattern that covers the entire arrangement (i.e. there are partitions that divide the arrangement into 'blocks' that are exactly the same). When depicted in a line, symmetric arrangements look like the same 'block' repeated more than once.

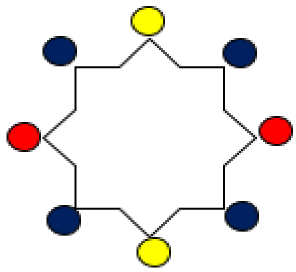


For this to be possible, we need to have the number of each kind of similar element (here colours) to be divisible by the number of blocks (as each block must contain the same number of each colour). Since this holds for all the colours, we take the set of common divisors of all the colours. Now there are 2 cases of the problem, which I will call 'simple' and 'compound' (yep too much math fractures my brain). In a 'simple' problem, the only common divisors of the colour numbers are 1 and the GCD (≠1) whereas in a 'compound' problem they may have more common divisors. Solving an example of each would give us further insight.

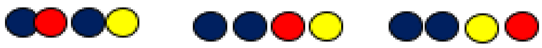
For the simple case, let us say we need to find the number of circular arrangements of 2 red, 4 blue and 2 yellow balls. We take the GCD or only common divisor of the elements (in this case 2) and that is the only possible number of blocks. In each block there will be 1 red, 2 blue and 1 yellow ball. We get the number of possible blocks here as $\frac{4!}{2!}$ (as there are 2 identical blue balls pun not intended)=12. This is the number of 'symmetric' lines generated by the given set (just copy the first block and append it at the end to get the line consisting of all the elements).

The number of possible symmetric circles is precisely the number of ways to arrange the first block in a circle. The reason for this is because cyclically permuting elements of the given block gives the same circular arrangement. This is verified below:





The two are essentially the same symmetric circular arrangement which are generated by the blocks RBYB and YBRB which are identical cyclically. Now how to find the number of symmetric circular arrangements? It is simply the number of ways to arrange a 'block' consisting of 1 red, 2 blue and 1 yellow ball in a circle. As there is no possible symmetry in this arrangement, we can use our 'naïve' formula here to get the answer, i.e. $\frac{3!}{2!}$=3. Indeed there are 3 possible symmetric circular arrangements, which are generated by the following blocks:

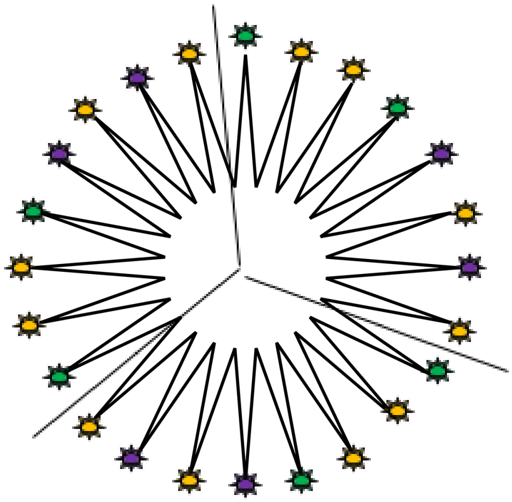


About the non symmetric arrangements? Well there are a total of 8!/4!2!2!=420 possible lines (again not intentional) out of which 12 are symmetric. The number of non symmetric lines is 408. They are divided into equivalence classes of 8 lines each of which generates one circle. So the answer is 408/8=51.

So our final answer is 51+3=54 possible arrangements.

Let us now try to solve a case of the compound problem. Let us say we need to find the possible circular arrangements of 6 green, 12 orange and 6 violet suns. The number of possible lines we can have from this set is $\frac{24!}{6!*12!*6!}$= something so big that we won't be doing the calculation explicitly.

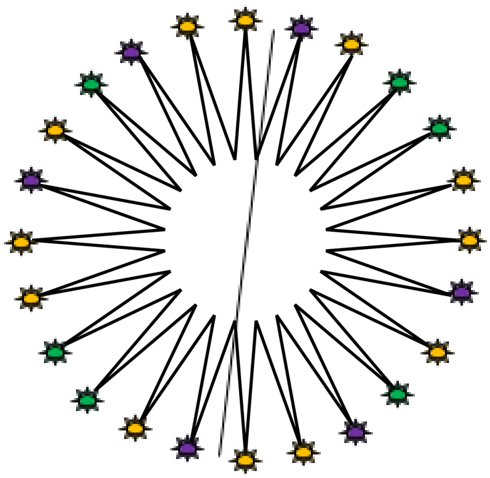
The set of common divisors of 6,12 and 6 is {2,3,6}. These are the possible numbers of blocks. We need to work with each of those. First let us choose 3. If there are 3 blocks, each will have 2 green, 4 orange and 2 violet. The number of circles generable with this gives us the number of 3-symmetric circular arrangements of the given set. But look carefully! This is the exact same as the 'simple' problem we just solved whose answer is 54.



An example of a 3-symmetric pattern

Next we work with 2 as the number of blocks. If there are 2 blocks each block will have 3 green, 6 orange and 3 violet suns. This is again a simple problem with symmetric arrangements generated from 3 blocks, each of which has 1 green, 2 orange and 1 violet sun and each symmetric circle corresponding to 4 lines. Solving this gives us our number of 2-symmetric patterns of the 'compound' problem:

An example of a 2-symmetric arrangement



We are done, except for the final step. How can a problem in combinatorics be complete, without an application of the Inclusion-Exclusion Principle? You guessed it write, 6-symmetric arrangements exist and have been counted twice, once as a 2-symmetric and once as a 3-symmetric arrangement. We get their number by the usual method, and subtract them from the total of the sum of 2 symmetric and 3 symmetric arrangements.

As for the non symmetric patterns, we know the number of non symmetric lines(refer to the 'simple' problem method, we can check how many lines each of the symmetric arrangements correspond to and subtract them from the total number of lines) and each circle must correspond to 24 lines. We therefore have our final answer, which is:

#non-symmetric patterns + #2-symmetric patterns + #3-symmetric patterns - # 6-symmetric patterns.

The formal way to solve this class of problems uses Burnside's lemma and Polya's enumeration theorem which I honestly have no clue about. I do not know if this problem has any applications anywhere and I just wrote on it because it looks cool and fun.